We approximate $E(k)$ for $k = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ using the first five terms:

$$E\left(\frac{1}{2}\right) \approx \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1}{2}\right)^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\left(\frac{1}{2}\right)^6 + \left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2\left(\frac{1}{2}\right)^8\right)$$

$$\approx 1.68517$$

The value given by a computer algebra system to seven places is $E\left(\frac{1}{2}\right) \approx 1.6856325$. ■

TABLE 1

Function $f(x)$	Maclaurin series	Converges to $f(x)$ for
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	All x
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	All x
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	All x
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$	$\lvert x\rvert < 1$
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots$	$\lvert x\rvert < 1$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$\lvert x\rvert < 1$ and $x = 1$
$\tan^{-1} x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$\lvert x\rvert \le 1$
$(1+x)^a$	$\sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots$	$\lvert x\rvert < 1$

10.7 SUMMARY

- *Taylor series* of $f(x)$ centered at $x = c$:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

The partial sum $T_k(x)$ is the kth Taylor polynomial.

- *Maclaurin series* ($c = 0$):

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

- If $f(x)$ is represented by a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ for $|x-c| < R$ with $R > 0$, then this power series is necessarily the Taylor series centered at $x = c$.
- A function $f(x)$ is represented by its Taylor series $T(x)$ if and only if the remainder $R_k(x) = f(x) - T_k(x)$ tends to zero as $k \to \infty$.
- Let $I = (c-R, c+R)$ with $R > 0$. Suppose that there exists $K > 0$ such that $|f^{(k)}(x)| < K$ for all $x \in I$ and all k. Then $f(x)$ is represented by its Taylor series on I; that is, $f(x) = T(x)$ for $x \in I$.
- A good way to find the Taylor series of a function is to start with known Taylor series and apply one of the operations: differentiation, integration, multiplication, or substitution.
- For any exponent a, the binomial expansion is valid for $|x| < 1$:

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 + \cdots + \binom{a}{n}x^n + \cdots$$

10.7 EXERCISES

Preliminary Questions

1. Determine $f(0)$ and $f'''(0)$ for a function $f(x)$ with Maclaurin series

$$T(x) = 3 + 2x + 12x^2 + 5x^3 + \cdots$$

2. Determine $f(-2)$ and $f^{(4)}(-2)$ for a function with Taylor series

$$T(x) = 3(x+2) + (x+2)^2 - 4(x+2)^3 + 2(x+2)^4 + \cdots$$

3. What is the easiest way to find the Maclaurin series for the function $f(x) = \sin(x^2)$?

4. Find the Taylor series for $f(x)$ centered at $c = 3$ if $f(3) = 4$ and $f'(x)$ has a Taylor expansion

$$f'(x) = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

5. Let $T(x)$ be the Maclaurin series of $f(x)$. Which of the following guarantees that $f(2) = T(2)$?

(a) $T(x)$ converges for $x = 2$.

(b) The remainder $R_k(2)$ approaches a limit as $k \to \infty$.

(c) The remainder $R_k(2)$ approaches zero as $k \to \infty$.

Exercises

1. Write out the first four terms of the Maclaurin series of $f(x)$ if

$$f(0) = 2, \quad f'(0) = 3, \quad f''(0) = 4, \quad f'''(0) = 12$$

2. Write out the first four terms of the Taylor series of $f(x)$ centered at $c = 3$ if

$$f(3) = 1, \quad f'(3) = 2, \quad f''(3) = 12, \quad f'''(3) = 3$$

In Exercises 3–18, find the Maclaurin series and find the interval on which the expansion is valid.

3. $f(x) = \dfrac{1}{1-2x}$

4. $f(x) = \dfrac{x}{1-x^4}$

5. $f(x) = \cos 3x$

6. $f(x) = \sin(2x)$

7. $f(x) = \sin(x^2)$

8. $f(x) = e^{4x}$

9. $f(x) = \ln(1-x^2)$

10. $f(x) = (1-x)^{-1/2}$

11. $f(x) = \tan^{-1}(x^2)$

12. $f(x) = x^2e^{x^2}$

13. $f(x) = e^{x-2}$

14. $f(x) = \dfrac{1-\cos x}{x}$

15. $f(x) = \ln(1-5x)$

16. $f(x) = (x^2+2x)e^x$

17. $f(x) = \sinh x$

18. $f(x) = \cosh x$

In Exercises 19–28, find the terms through degree four of the Maclaurin series of $f(x)$. Use multiplication and substitution as necessary.

19. $f(x) = e^x \sin x$

20. $f(x) = e^x \ln(1-x)$

21. $f(x) = \dfrac{\sin x}{1-x}$

22. $f(x) = \dfrac{1}{1+\sin x}$

23. $f(x) = (1+x)^{1/4}$

24. $f(x) = (1+x)^{-3/2}$

25. $f(x) = e^x \tan^{-1} x$

26. $f(x) = \sin(x^3 - x)$

27. $f(x) = e^{\sin x}$

28. $f(x) = e^{(e^x)}$

In Exercises 29–38, find the Taylor series centered at c and find the interval on which the expansion is valid.

29. $f(x) = \dfrac{1}{x}, \quad c = 1$

30. $f(x) = e^{3x}, \quad c = -1$

31. $f(x) = \frac{1}{1-x}$, $c = 5$

32. $f(x) = \sin x$, $c = \frac{\pi}{2}$

33. $f(x) = x^4 + 3x - 1$, $c = 2$

34. $f(x) = x^4 + 3x - 1$, $c = 0$

35. $f(x) = \frac{1}{x^2}$, $c = 4$

36. $f(x) = \sqrt{x}$, $c = 4$

37. $f(x) = \frac{1}{1-x^2}$, $c = 3$

38. $f(x) = \frac{1}{3x-2}$, $c = -1$

39. Use the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to find the Maclaurin series for $\cos^2 x$.

40. Show that for $|x| < 1$,

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Hint: Recall that $\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$.

41. Use the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to show that

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

for $|x| < 1$. What can you conclude by comparing this result with that of Exercise 40?

42. Differentiate the Maclaurin series for $\frac{1}{1-x}$ twice to find the Maclaurin series of $\frac{1}{(1-x)^3}$.

43. Show, by integrating the Maclaurin series for $f(x) = \frac{1}{\sqrt{1-x^2}}$, that for $|x| < 1$,

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \frac{x^{2n+1}}{2n+1}$$

44. Use the first five terms of the Maclaurin series in Exercise 43 to approximate $\sin^{-1} \frac{1}{2}$. Compare the result with the calculator value.

45. How many terms of the Maclaurin series of $f(x) = \ln(1+x)$ are needed to compute ln 1.2 to within an error of at most 0.0001? Make the computation and compare the result with the calculator value.

46. Show that

$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$$

converges to zero. How many terms must be computed to get within 0.01 of zero?

47. Use the Maclaurin expansion for e^{-t^2} to express the function $F(x) = \int_0^x e^{-t^2}\,dt$ as an alternating power series in x (Figure 4).

(a) How many terms of the Maclaurin series are needed to approximate the integral for $x = 1$ to within an error of at most 0.001?

(b) CAS Carry out the computation and check your answer using a computer algebra system.

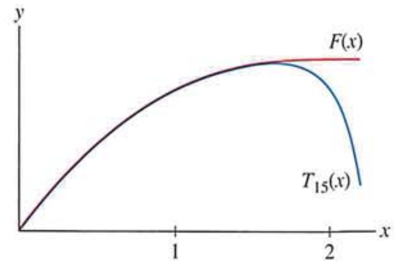

FIGURE 4 The Maclaurin polynomial $T_{15}(x)$ for $F(t) = \int_0^x e^{-t^2}\,dt$.

48. Let $F(x) = \int_0^x \frac{\sin t\,dt}{t}$. Show that

$$F(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

Evaluate $F(1)$ to three decimal places.

In Exercises 49–52, express the definite integral as an infinite series and find its value to within an error of at most 10^{-4}.

49. $\int_0^1 \cos(x^2)\,dx$

50. $\int_0^1 \tan^{-1}(x^2)\,dx$

51. $\int_0^1 e^{-x^3}\,dx$

52. $\int_0^1 \frac{dx}{\sqrt{x^4+1}}$

In Exercises 53–56, express the integral as an infinite series.

53. $\int_0^x \frac{1-\cos(t)}{t}\,dt$, for all x

54. $\int_0^x \frac{t - \sin t}{t}\,dt$, for all x

55. $\int_0^x \ln(1+t^2)\,dt$, for $|x| < 1$

56. $\int_0^x \frac{dt}{\sqrt{1-t^4}}$, for $|x| < 1$

57. Which function has Maclaurin series $\sum_{n=0}^{\infty} (-1)^n 2^n x^n$?

58. Which function has Maclaurin series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{3^{k+1}}(x-3)^k?$$

For which values of x is the expansion valid?

In Exercises 59–62, use Theorem 2 to prove that the $f(x)$ is represented by its Maclaurin series for all x.

59. $f(x) = \sin(x/2) + \cos(x/3)$

60. $f(x) = e^{-x}$

61. $f(x) = \sinh x$

62. $f(x) = (1+x)^{100}$

In Exercises 63–66, find the functions with the following Maclaurin series (refer to Table 1 on page 599).

63. $1 + x^3 + \dfrac{x^6}{2!} + \dfrac{x^9}{3!} + \dfrac{x^{12}}{4!} + \cdots$

64. $1 - 4x + 4^2x^2 - 4^3x^3 + 4^4x^4 - 4^5x^5 + \cdots$

65. $1 - \dfrac{5^3x^3}{3!} + \dfrac{5^5x^5}{5!} - \dfrac{5^7x^7}{7!} + \cdots$

66. $x^4 - \dfrac{x^{12}}{3} + \dfrac{x^{20}}{5} - \dfrac{x^{28}}{7} + \cdots$

In Exercises 67 and 68, let

$$f(x) = \frac{1}{(1-x)(1-2x)}$$

67. Find the Maclaurin series of $f(x)$ using the identity

$$f(x) = \frac{2}{1-2x} - \frac{1}{1-x}$$

68. Find the Taylor series for $f(x)$ at $c = 2$. *Hint:* Rewrite the identity of Exercise 67 as

$$f(x) = \frac{2}{-3-2(x-2)} - \frac{1}{-1-(x-2)}$$

69. When a voltage V is applied to a series circuit consisting of a resistor R and an inductor L, the current at time t is

$$I(t) = \left(\frac{V}{R}\right)\left(1 - e^{-Rt/L}\right)$$

Expand $I(t)$ in a Maclaurin series. Show that $I(t) \approx \dfrac{Vt}{L}$ for small t.

70. Use the result of Exercise 69 and your knowledge of alternating series to show that

$$\frac{Vt}{L}\left(1 - \frac{R}{2L}t\right) \le I(t) \le \frac{Vt}{L} \qquad \text{(for all } t\text{)}$$

71. Find the Maclaurin series for $f(x) = \cos(x^3)$ and use it to determine $f^{(6)}(0)$.

72. Find $f^{(7)}(0)$ and $f^{(8)}(0)$ for $f(x) = \tan^{-1} x$ using the Maclaurin series.

73. Use substitution to find the first three terms of the Maclaurin series for $f(x) = e^{x^{20}}$. How does the result show that $f^{(k)}(0) = 0$ for $1 \le k \le 19$?

74. Use the binomial series to find $f^{(8)}(0)$ for $f(x) = \sqrt{1-x^2}$.

75. Does the Maclaurin series for $f(x) = (1+x)^{3/4}$ converge to $f(x)$ at $x = 2$? Give numerical evidence to support your answer.

76. Explain the steps required to verify that the Maclaurin series for $f(x) = e^x$ converges to $f(x)$ for all x.

77. GU Let $f(x) = \sqrt{1+x}$.

(a) Use a graphing calculator to compare the graph of f with the graphs of the first five Taylor polynomials for f. What do they suggest about the interval of convergence of the Taylor series?

(b) Investigate numerically whether or not the Taylor expansion for f is valid for $x = 1$ and $x = -1$.

78. Use the first five terms of the Maclaurin series for the elliptic function $E(k)$ to estimate the period T of a 1-meter pendulum released at an angle $\theta = \frac{\pi}{4}$ (see Example 11).

79. Use Example 11 and the approximation $\sin x \approx x$ to show that the period T of a pendulum released at an angle θ has the following second-order approximation:

$$T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{\theta^2}{16}\right)$$

In Exercises 80–83, find the Maclaurin series of the function and use it to calculate the limit.

80. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$

81. $\displaystyle\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$

82. $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x - x\cos x - \frac{1}{6}x^3}{x^5}$

83. $\displaystyle\lim_{x\to 0} \left(\frac{\sin(x^2)}{x^4} - \frac{\cos x}{x^2}\right)$

Further Insights and Challenges

84. In this exercise we show that the Maclaurin expansion of $f(x) = \ln(1+x)$ is valid for $x = 1$.

(a) Show that for all $x \neq -1$,

$$\frac{1}{1+x} = \sum_{n=0}^{N} (-1)^n x^n + \frac{(-1)^{N+1}x^{N+1}}{1+x}$$

(b) Integrate from 0 to 1 to obtain

$$\ln 2 = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n} + (-1)^{N+1}\int_0^1 \frac{x^{N+1}\,dx}{1+x}$$

(c) Verify that the integral on the right tends to zero as $N \to \infty$ by showing that it is smaller than $\int_0^1 x^{N+1}\,dx$.

(d) Prove the formula

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

85. Let $g(t) = \dfrac{1}{1+t^2} - \dfrac{t}{1+t^2}$.

(a) Show that $\displaystyle\int_0^1 g(t)\,dt = \frac{\pi}{4} - \frac{1}{2}\ln 2$.

(b) Show that $g(t) = 1 - t - t^2 + t^3 + t^4 - t^5 - t^6 + \cdots$.

(c) Evaluate $S = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$.