5.1 SUMMARY

Power Sums

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2} = \frac{N^2}{2} + \frac{N}{2}$$

$$\sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6} = \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}$$

$$\sum_{j=1}^{N} j^3 = \frac{N^2(N+1)^2}{4} = \frac{N^4}{4} + \frac{N^3}{2} + \frac{N^2}{4}$$

- Approximations to the area under the graph of f over the interval $[a, b]$ $\left(\Delta x = \frac{b-a}{N}, x_j = a + j\Delta x\right)$:

$$R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \Delta x\big(f(x_1) + f(x_2) + \cdots + f(x_N)\big)$$

$$L_N = \Delta x \sum_{j=0}^{N-1} f(x_j) = \Delta x\big(f(x_0) + f(x_1) + \cdots + f(x_{N-1})\big)$$

$$M_N = \Delta x \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right)$$

$$= \Delta x \left(f\left(\frac{x_0 + x_1}{2}\right) + \cdots + f\left(\frac{x_{N-1} + x_N}{2}\right)\right)$$

- If f is continuous on $[a, b]$, then the endpoint and midpoint approximations approach one and the same limit L:

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = L$$

- If $f(x) \geq 0$ on $[a, b]$, we take L as the definition of the area under the graph of $y = f(x)$ over $[a, b]$.

5.1 EXERCISES

Preliminary Questions

1. What are the right and left endpoints if $[2, 5]$ is divided into six subintervals?

2. The interval $[1, 5]$ is divided into eight subintervals.

(a) What is the left endpoint of the last subinterval?

(b) What are the right endpoints of the first two subintervals?

3. Which of the following pairs of sums are *not* equal?

(a) $\sum_{i=1}^{4} i, \quad \sum_{\ell=1}^{4} \ell$

(b) $\sum_{j=1}^{4} j^2, \quad \sum_{k=2}^{5} k^2$

(c) $\sum_{j=1}^{4} j, \quad \sum_{i=2}^{5} (i-1)$

(d) $\sum_{i=1}^{4} i(i+1), \quad \sum_{j=2}^{5} (j-1)j$

4. Explain: $\sum_{j=1}^{100} j = \sum_{j=0}^{100} j$ but $\sum_{j=1}^{100} 1$ is not equal to $\sum_{j=0}^{100} 1$.

5. Explain why $L_{100} \geq R_{100}$ for $f(x) = x^{-2}$ on $[3, 7]$.

Exercises

1. Figure 15 shows the velocity of an object over a 3-minute (min) interval. Determine the distance traveled over the intervals $[0, 3]$ and $[1, 2.5]$ (remember to convert from kilometers per hour to kilometers per minute).

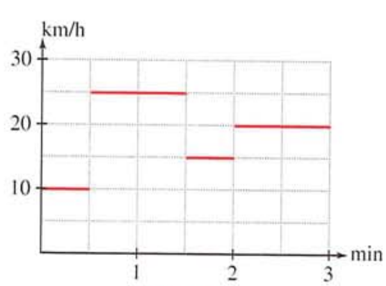


FIGURE 15

2. An ostrich (Figure 16) runs with velocity 20 km/h for 2 minutes (min), 12 km/h for 3 min, and 40 km/h for another minute. Compute the total distance traveled and indicate with a graph how this quantity can be interpreted as an area.

FIGURE 16 Ostriches can reach speeds as high as 70 km/h. (© Daryl Balfour/Gallo Images/Alamy)

3. A rainstorm hit Portland, Maine, in October 1996, resulting in record rainfall. The rainfall rate $R(t)$ on October 21 is recorded, in centimeters per hour, in the following table, where t is the number of hours since midnight. Compute the total rainfall during this 24-hour period and indicate on a graph how this quantity can be interpreted as an area.

t (h)	0–2	2–4	4–9	9–12	12–20	20–24
$R(t)$ (cm/h)	0.5	0.3	1.0	2.5	1.5	0.6

4. The velocity of an object is $v(t) = 12t$ m/s. Use Eq. (2) and geometry to find the distance traveled over the time intervals $[0, 2]$ and $[2, 5]$.

5. Compute R_5 and L_5 over $[0, 1]$ using the following values:

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	50	48	46	44	42	40

6. Compute R_6, L_6, and M_3 to estimate the distance traveled over $[0, 3]$ if the velocity at half-second intervals is as follows:

t (s)	0	0.5	1	1.5	2	2.5	3
v (m/s)	0	12	18	25	20	14	20

7. Let $f(x) = 2x + 3$.

(a) Compute R_6 and L_6 over $[0, 3]$.

(b) Use geometry to find the exact area A and compute the errors $|A - R_6|$ and $|A - L_6|$ in the approximations.

8. Repeat Exercise 7 for $f(x) = 20 - 3x$ over $[2, 4]$.

9. Calculate R_3 and L_3 for $f(x) = x^2 - x + 4$ over $[1, 4]$. Then sketch the graph of f and the rectangles that make up each approximation. Is the area under the graph larger or smaller than R_3? Is it larger or smaller than L_3?

10. Let $f(x) = \sqrt{x^2 + 1}$ and $\Delta x = \frac{1}{3}$. Sketch the graph of f and draw the right-endpoint rectangles whose area is represented by the sum $\sum_{i=1}^{6} f(1 + i\Delta x)\Delta x$.

11. Estimate R_3, M_3, and L_6 over $[0, 1.5]$ for the function in Figure 17.

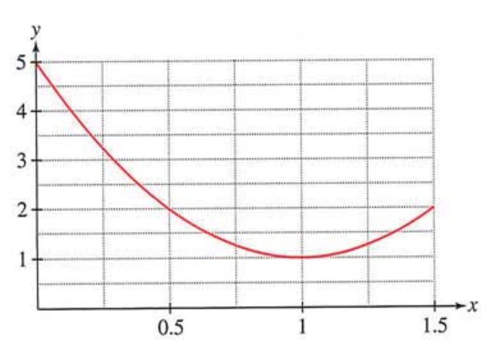


FIGURE 17

12. Calculate the area of the shaded rectangles in Figure 18. Which approximation do these rectangles represent?

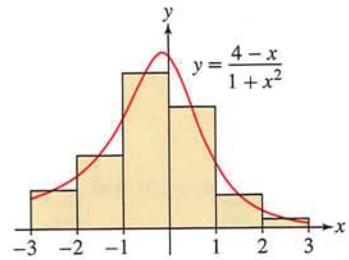


FIGURE 18

13. Let $f(x) = x^2$.

(a) Sketch the function over the interval $[0, 2]$ and the rectangles corresponding to L_4. Calculate the area contained within them.

(b) Sketch the function over the interval $[0, 2]$ again but with the rectangles corresponding to R_4. Calculate the area contained within them.

(c) Make a conclusion about the area under the curve $f(x) = x^2$ over the interval $[0, 2]$.

14. Let $f(x) = \sqrt{x}$.

(a) Sketch the function over the interval $[0, 4]$ and the rectangles corresponding to L_4. Calculate the area contained within them.

(b) Sketch the function over the interval $[0, 4]$ again but with the rectangles corresponding to R_4. Calculate the area contained within them.

(c) Make a conclusion about the area under the curve $f(x) = \sqrt{x}$ over the interval $[0, 4]$.

In Exercises 15–22, calculate the approximation for the given function and interval.

15. R_3, $f(x) = 7 - x$, $[3, 5]$

16. L_6, $f(x) = \sqrt{6x + 2}$, $[1, 3]$

17. M_6, $f(x) = 4x + 3$, $[5, 8]$

18. R_5, $f(x) = x^2 + x$, $[-1, 1]$

19. M_5, $f(x) = \ln x$, $[1, 3]$

20. M_4, $f(x) = \sqrt{x}$, $[3, 5]$

21. L_4, $f(x) = \cos^2 x$, $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

22. L_6, $f(x) = x^2 + 3|x|$, $[-2, 1]$

In Exercises 23–28, write the sum in summation notation.

23. $4^7 + 5^7 + 6^7 + 7^7 + 8^7$

24. $(2^2 + 2) + (3^2 + 3) + (4^2 + 4) + (5^2 + 5)$

25. $(2^2 + 2) + (2^3 + 2) + (2^4 + 2) + (2^5 + 2)$

26. $\sqrt{1 + 1^3} + \sqrt{2 + 2^3} + \cdots + \sqrt{n + n^3}$

27. $\dfrac{1}{2 \cdot 3} + \dfrac{2}{3 \cdot 4} + \cdots + \dfrac{n}{(n + 1)(n + 2)}$

28. $e^{\pi} + e^{\pi/2} + e^{\pi/3} + \cdots + e^{\pi/n}$

29. Calculate the sums:

(a) $\sum_{i=1}^{5} 9$ **(b)** $\sum_{i=0}^{5} 4$ **(c)** $\sum_{k=2}^{4} k^3$

30. Calculate the sums:

(a) $\sum_{j=3}^{4} \sin\left(j\frac{\pi}{2}\right)$ **(b)** $\sum_{k=3}^{5} \frac{1}{k - 1}$ **(c)** $\sum_{j=0}^{2} 3^{j-1}$

31. Let $b_1 = 4$, $b_2 = 1$, $b_3 = 2$, and $b_4 = -4$. Calculate:

(a) $\sum_{i=2}^{4} b_i$ **(b)** $\sum_{j=1}^{2} (2^{b_j} - b_j)$ **(c)** $\sum_{k=1}^{3} kb_k$

32. Assume that $a_1 = -5$, $\sum_{i=1}^{10} a_i = 20$, and $\sum_{i=1}^{10} b_i = 7$. Calculate:

(a) $\sum_{i=1}^{10} (4a_i + 3)$ **(b)** $\sum_{i=2}^{10} a_i$ **(c)** $\sum_{i=1}^{10} (2a_i - 3b_i)$

33. Calculate $\sum_{j=101}^{200} j$. *Hint:* Write as a difference of two sums and use formula (3).

34. Calculate $\sum_{j=1}^{30} (2j+1)^2$. *Hint:* Expand and use formulas (3)–(4).

In Exercises 35–42, use linearity and formulas (3)–(5) to rewrite and evaluate the sums.

35. $\sum_{j=1}^{20} 8j^3$

36. $\sum_{k=1}^{30} (4k - 3)$

37. $\sum_{n=51}^{150} n^2$

38. $\sum_{k=101}^{200} k^3$

39. $\sum_{j=0}^{50} j(j-1)$

40. $\sum_{j=2}^{30} \left(6j + \frac{4j^2}{3}\right)$

41. $\sum_{m=1}^{30} (4-m)^3$

42. $\sum_{m=1}^{20} \left(5 + \frac{3m}{2}\right)^2$

In Exercises 43–46, use formulas (3)–(5) to evaluate the limit.

43. $\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N^2}$

44. $\lim_{N\to\infty} \sum_{j=1}^{N} \frac{j^3}{N^4}$

45. $\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3}$

46. $\lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{i^3}{N^4} - \frac{20}{N}\right)$

In Exercises 47–52, calculate the limit for the given function and interval. Verify your answer by using geometry.

47. $\lim_{N\to\infty} R_N$, $f(x) = 9x$, $[0, 2]$

48. $\lim_{N\to\infty} R_N$, $f(x) = 3x + 6$, $[1, 4]$

49. $\lim_{N\to\infty} L_N$, $f(x) = \frac{1}{2}x + 2$, $[0, 4]$

50. $\lim_{N\to\infty} L_N$, $f(x) = 4x - 2$, $[1, 3]$

51. $\lim_{N\to\infty} M_N$, $f(x) = x$, $[0, 2]$

52. $\lim_{N\to\infty} M_N$, $f(x) = 12 - 4x$, $[2, 6]$

53. Show, for $f(x) = 3x^2 + 4x$ over $[0, 2]$, that

$$R_N = \frac{2}{N} \sum_{j=1}^{N} \left(\frac{12j^2}{N^2} + \frac{8j}{N}\right)$$

Then evaluate $\lim_{N\to\infty} R_N$.

54. Show, for $f(x) = 3x^3 - x^2$ over $[1, 5]$, that

$$R_N = \frac{4}{N} \sum_{j=1}^{N} \left(\frac{192j^3}{N^3} + \frac{128j^2}{N^2} + \frac{28j}{N} + 2\right)$$

Then evaluate $\lim_{N\to\infty} R_N$.

In Exercises 55–62, find a formula for R_N and compute the area under the graph as a limit.

55. $f(x) = x^2$, $[0, 1]$

56. $f(x) = x^2$, $[-1, 5]$

57. $f(x) = 6x^2 - 4$, $[2, 5]$

58. $f(x) = x^2 + 7x$, $[6, 11]$

59. $f(x) = x^3 - x$, $[0, 2]$

60. $f(x) = 2x^3 + x^2$, $[-2, 2]$

61. $f(x) = 2x + 1$, $[a, b]$ (a, b constants with $a < b$)

62. $f(x) = x^2$, $[a, b]$ (a, b constants with $a < b$)

In Exercises 63–66, describe the area represented by the limits.

63. $\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\frac{j}{N}\right)^4$

64. $\lim_{N\to\infty} \frac{3}{N} \sum_{j=1}^{N} \left(2 + \frac{3j}{N}\right)^4$

65. $\lim_{N\to\infty} \frac{5}{N} \sum_{j=0}^{N-1} e^{-2+5j/N}$

66. $\lim_{N\to\infty} \frac{\pi}{2N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} - \frac{\pi}{4N} + \frac{j\pi}{2N}\right)$

In Exercises 67–72, express the area under the graph as a limit using the approximation indicated (in summation notation), but do not evaluate.

67. R_N, $f(x) = \sin x$ over $[0, \pi]$

68. R_N, $f(x) = x^{-1}$ over $[1, 7]$

69. L_N, $f(x) = \sqrt{2x+1}$ over $[7, 11]$

70. L_N, $f(x) = \cos x$ over $\left[\frac{\pi}{8}, \frac{\pi}{4}\right]$

71. M_N, $f(x) = \tan x$ over $\left[\frac{1}{2}, 1\right]$

72. M_N, $f(x) = x^{-2}$ over $[3, 5]$

73. Evaluate $\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left(\frac{j}{N}\right)^2}$ by interpreting it as the area of part of a familiar geometric figure.

In Exercises 74–76, let $f(x) = x^2$ and let R_N, L_N, and M_N be the approximations for the interval $[0, 1]$.

74. Show that $R_N = \frac{1}{3} + \frac{1}{2N} + \frac{1}{6N^2}$. Interpret the quantity $\frac{1}{2N} + \frac{1}{6N^2}$ as the area of a region.

75. Show that

$$L_N = \frac{1}{3} - \frac{1}{2N} + \frac{1}{6N^2}, \qquad M_N = \frac{1}{3} - \frac{1}{12N^2}$$

Then rank the three approximations R_N, L_N, and M_N in order of increasing accuracy (use Exercise 74).

76. For each of R_N, L_N, and M_N, find the smallest integer N for which the error is less than 0.001.

In Exercises 77–82, use the Graphical Insight on page 264 to obtain bounds on the area.

77. Let A be the area under $f(x) = \sqrt{x}$ over $[0, 1]$. Prove that $0.51 \le A \le 0.77$ by computing R_4 and L_4. Explain your reasoning.

78. Use R_5 and L_5 to show that the area A under $y = x^{-2}$ over $[10, 13]$ satisfies $0.0218 \le A \le 0.0244$.

79. Use R_4 and L_4 to show that the area A under the graph of $y = \sin x$ over $\left[0, \frac{\pi}{2}\right]$ satisfies $0.79 \le A \le 1.19$.

80. Show that the area A under $f(x) = x^{-1}$ over $[1, 8]$ satisfies

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \le A \le 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$

81. CAS Show that the area A under $y = x^{1/4}$ over $[0, 1]$ satisfies $L_N \le A \le R_N$ for all N. Use a computer algebra system to calculate L_N and R_N for $N = 100$ and 200, and determine A to two decimal places.

82. CAS Show that the area A under $y = 4/(x^2 + 1)$ over $[0, 1]$ satisfies $R_N \le A \le L_N$ for all N. Determine A to at least three decimal places using a computer algebra system. Can you guess the exact value of A?

83. In this exercise, we evaluate the area A under the graph of $y = e^x$ over $[0, 1]$ [Figure 19(A)] using the formula for a geometric sum (valid for $r \ne 1$):

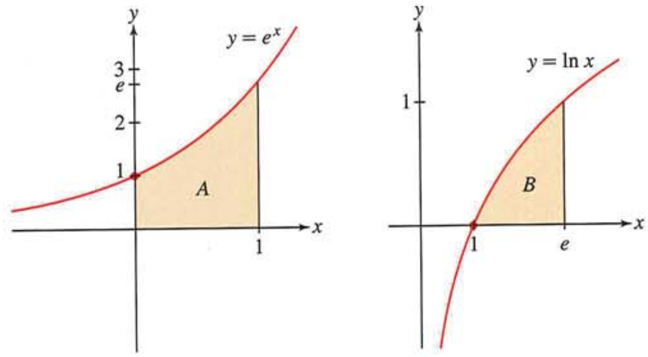
$$1 + r + r^2 + \cdots + r^{N-1} = \sum_{j=0}^{N-1} r^j = \frac{r^N - 1}{r - 1} \qquad \boxed{8}$$

(a) Show that $L_N = \dfrac{1}{N} \displaystyle\sum_{j=0}^{N-1} e^{j/N}$.

(b) Apply Eq. (8) with $r = e^{1/N}$ to prove $L_N = \dfrac{e - 1}{N(e^{1/N} - 1)}$.

(c) Compute $A = \displaystyle\lim_{N\to\infty} L_N$ using L'Hôpital's Rule.

84. Use the result of Exercise 83 to show that the area B under the graph of $f(x) = \ln x$ over $[1, e]$ is equal to 1. *Hint:* Relate B in Figure 19(B) to the area A computed in Exercise 83.



(A) (B)

FIGURE 19

Further Insights and Challenges

85. Although the accuracy of R_N generally improves as N increases, this need not be true for small values of N. Draw the graph of a positive continuous function f on an interval such that R_1 is closer than R_2 to the exact area under the graph. Can such a function be monotonic?

86. Draw the graph of a positive continuous function on an interval such that R_2 and L_2 are both smaller than the exact area under the graph. Can such a function be monotonic?

87. Explain graphically: *The endpoint approximations are less accurate when $f'(x)$ is large.*

88. Prove that for any function f on $[a, b]$,

$$R_N - L_N = \frac{b-a}{N}(f(b) - f(a)) \qquad \boxed{9}$$

89. In this exercise, we prove that $\displaystyle\lim_{N\to\infty} R_N$ and $\displaystyle\lim_{N\to\infty} L_N$ exist and are equal if f is increasing [the case of f decreasing is similar]. We use the concept of a least upper bound discussed in Appendix B.

(a) Explain with a graph why $L_N \le R_M$ for all $N, M \ge 1$.

(b) By (a), the sequence $\{L_N\}$ is bounded, so it has a least upper bound L. By definition, L is the smallest number such that $L_N \le L$ for all N. Show that $L \le R_M$ for all M.

(c) According to (b), $L_N \le L \le R_N$ for all N. Use Eq. (9) to show that $\displaystyle\lim_{N\to\infty} L_N = L$ and $\displaystyle\lim_{N\to\infty} R_N = L$.

90. Use Eq. (9) to show that if f is positive and monotonic, then the area A under its graph over $[a, b]$ satisfies

$$|R_N - A| \le \frac{b-a}{N}|f(b) - f(a)| \qquad \boxed{10}$$

In Exercises 91–92, use Eq. (10) to find a value of N such that $|R_N - A| < 10^{-4}$ for the given function and interval.

91. $f(x) = \sqrt{x}$, $[1, 4]$

92. $f(x) = \sqrt{9 - x^2}$, $[0, 3]$

93. Prove that if f is positive and monotonic, then M_N lies between R_N and L_N and is closer to the actual area under the graph than both R_N and L_N. *Hint:* In the case that f is increasing, Figure 20 shows that the part of the error in R_N due to the ith rectangle is the sum of the areas $A + B + D$, and for M_N it is $|B - E|$. On the other hand, $A \ge E$.

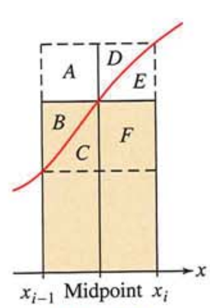


FIGURE 20