5.3 SUMMARY

- F is called an *antiderivative* of f if $F'(x) = f(x)$.
- Any two antiderivatives of f on an interval (a, b) differ by a constant.
- The general antiderivative is denoted by the indefinite integral:

$$\int f(x)\,dx = F(x) + C$$

- Integration formulas:

$$\int 0\,dx = C$$

$$\int k\,dx = kx + C \qquad (k \neq 0)$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C \qquad (k \neq 0)$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C \qquad (k \neq 0)$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C \qquad (k \neq 0)$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

- To solve an initial value problem $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$, first find the general antiderivative $y = F(x) + C$. Then determine C using the initial condition $F(x_0) + C = y_0$.

5.3 EXERCISES

Preliminary Questions

1. Find an antiderivative of the function $f(x) = 0$.

2. Is there a difference between finding the general antiderivative of a function f and evaluating $\int f(x)\,dx$?

3. Jacques was told that f and g have the same derivative, and he wonders whether $f(x) = g(x)$. Does Jacques have sufficient information to answer his question?

4. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Which of the following statements are true? Explain.

(a) If $f = g$, then $F = G$.

(b) If F and G differ by a constant, then $f = g$.

(c) If f and g differ by a constant, then $F = G$.

5. Is $y = x$ a solution of the following initial value problem?

$$\frac{dy}{dx} = 1, \qquad y(0) = 1$$

Exercises

In Exercises 1–8, find the general antiderivative of f and check your answer by differentiating.

1. $f(x) = 18x^2$

2. $f(x) = x^{-3/5}$

3. $f(x) = 2x^4 - 24x^2 + 12x^{-1}$

4. $f(x) = 9x + 15x^{-2}$

5. $f(x) = 2\cos x - 9\sin x$

6. $f(x) = 4x^7 - 3\cos x$

7. $f(x) = 12e^x - 5x^{-2}$

8. $f(x) = e^x - 4\sin x$

9. Match functions (a)–(d) with their antiderivatives (i)–(iv).

(a) $f(x) = \sin x$	(i) $F(x) = \cos(1 - x)$
(b) $f(x) = x\sin(x^2)$	(ii) $F(x) = -\cos x$
(c) $f(x) = \sin(1 - x)$	(iii) $F(x) = -\frac{1}{2}\cos(x^2)$
(d) $f(x) = x\sin x$	(iv) $F(x) = \sin x - x\cos x$

In Exercises 10–39, evaluate the indefinite integral.

10. $\int (9x+2)\,dx$

11. $\int (4-18x)\,dx$

12. $\int x^{-3}\,dx$

13. $\int t^{-6/11}\,dt$

14. $\int (5t^3 - t^{-3})\,dt$

15. $\int (18t^5 - 10t^4 - 28t)\,dt$

16. $\int 14s^{9/5}\,ds$

17. $\int (z^{-4/5} - z^{2/3} + z^{5/4})\,dz$

18. $\int \frac{3}{2}\,dx$

19. $\int \frac{1}{\sqrt[3]{x}}\,dx$

20. $\int \frac{dx}{x^{4/3}}$

21. $\int \frac{36\,dt}{t^3}$

22. $\int x(x^2-4)\,dx$

23. $\int (t^{1/2}+1)(t+1)\,dt$

24. $\int \frac{12-z}{\sqrt{z}}\,dz$

25. $\int \frac{x^3+3x-4}{x^2}\,dx$

26. $\int \left(\frac{1}{3}\sin x - \frac{1}{4}\cos x\right)dx$

27. $\int 12\sec x\tan x\,dx$

28. $\int (\theta + \sec^2\theta)\,d\theta$

29. $\int (\csc t\cot t)\,dt$

30. $\int \sin(7x)\,dx$

31. $\int \sec^2(-3\theta)\,d\theta$

32. $\int (\theta - \cos(-\theta))\,d\theta$

33. $\int 25\sec^2(3z)\,dz$

34. $\int \sec x\tan x\,dx$

35. $\int \left(\cos(3\theta) - \frac{1}{2}\sec^2\left(\frac{\theta}{4}\right)\right)d\theta$

36. $\int \left(\frac{4}{x} - e^x\right)dx$

37. $\int (3e^{5x})\,dx$

38. $\int e^{3t-4}\,dt$

39. $\int (8x - 4e^{5-2x})\,dx$

40. In Figure 3, is graph (A) or graph (B) the graph of an antiderivative of $y = f(x)$?

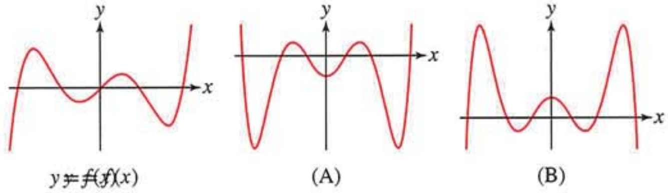

FIGURE 3

41. In Figure 4, which of graphs (A), (B), and (C) is *not* the graph of an antiderivative of $y = f(x)$? Explain.

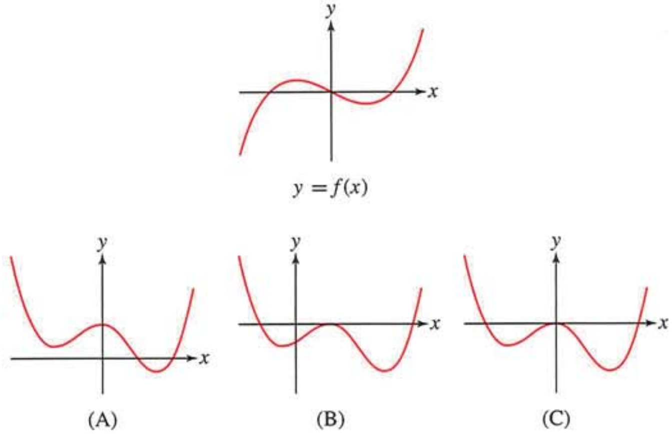

FIGURE 4

42. Show that $F(x) = \frac{1}{3}(x+13)^3$ is an antiderivative of $f(x) = (x+13)^2$.

In Exercises 43–46, verify by differentiation.

43. $\int (x+13)^6\,dx = \frac{1}{7}(x+13)^7 + C$

44. $\int (x+13)^{-5}\,dx = -\frac{1}{4}(x+13)^{-4} + C$

45. $\int (4x+13)^2\,dx = \frac{1}{12}(4x+13)^3 + C$

46. $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + C \qquad (\text{for } n \neq -1)$

In Exercises 47–62, solve the initial value problem.

47. $\frac{dy}{dx} = x^3,\ y(0) = 4$

48. $\frac{dy}{dt} = 3 - 2t,\ y(0) = -5$

49. $\frac{dy}{dt} = 2t + 9t^2,\ y(1) = 2$

50. $\frac{dy}{dx} = 8x^3 + 3x^2,\ y(2) = 0$

51. $\frac{dy}{dt} = \sqrt{t},\ y(1) = 1$

52. $\frac{dz}{dt} = t^{-3/2},\ z(4) = -1$

53. $\frac{dy}{dx} = (3x+2)^3,\ y(0) = 1$

54. $\frac{dy}{dt} = (4t+3)^{-2},\ y(1) = 0$

55. $\frac{dy}{dx} = \sin x,\ y\left(\frac{\pi}{2}\right) = 1$

56. $\frac{dy}{dz} = \sin 2z,\ y\left(\frac{\pi}{4}\right) = 4$

57. $\frac{dy}{dx} = \cos 5x,\ y(\pi) = 3$

58. $\frac{dy}{dx} = \sec^2 3x,\ y\left(\frac{\pi}{4}\right) = 2$

59. $\frac{dy}{dx} = e^x,\ y(2) = 0$

60. $\frac{dy}{dt} = e^{-t},\ y(0) = 0$

61. $\frac{dy}{dt} = 9e^{12-3t},\ y(4) = 7$

62. $\frac{dy}{dt} = t + 2e^{t-9},\ y(9) = 4$

In Exercises 63–69, first find f' and then find f.

63. $f''(x) = 12x,\quad f'(0) = 1,\quad f(0) = 2$

64. $f''(x) = x^3 - 2x, \quad f'(1) = 0, \quad f(1) = 2$

65. $f''(x) = x^3 - 2x + 1, \quad f'(0) = 1, \quad f(0) = 0$

66. $f''(x) = x^3 - 2x + 1, \quad f'(1) = 0, \quad f(1) = 4$

67. $f''(t) = t^{-3/2}, \quad f'(4) = 1, \quad f(4) = 4$

68. $f''(\theta) = \cos\theta, \quad f'\left(\frac{\pi}{2}\right) = 1, \quad f\left(\frac{\pi}{2}\right) = 6$

69. $f''(t) = t - \cos t, \quad f'(0) = 2, \quad f(0) = -2$

70. Show that $F(x) = \tan^2 x$ and $G(x) = \sec^2 x$ have the same derivative. What can you conclude about the relation between F and G? Verify this conclusion directly.

71. A particle located at the origin at $t = 1$ s moves along the x-axis with velocity $v(t) = (6t^2 - t)$ m/s. State the differential equation with its initial condition satisfied by the position $s(t)$ of the particle, and find $s(t)$.

72. A particle moves along the x-axis with velocity $v(t) = (6t^2 - t)$ m/s. Find the particle's position $s(t)$, assuming that $s(2) = 4$ m.

73. A water balloon is dropped from a high building. It falls for 5 s before hitting the ground. Determine the velocity it is traveling when it is about to hit the ground, assuming an acceleration due to gravity of -9.8 m/s^2 and no wind resistance.

74. A hammer is dropped and it falls for 2 s before hitting the ground. Determine how far it falls, assuming an acceleration due to gravity of -9.8 m/s^2 and no wind resistance.

75. A mass oscillates at the end of a spring. Let $s(t)$ be the displacement of the mass from the equilibrium position at time t. Assuming that the mass is located at the origin at $t = 0$ and has velocity $v(t) = \sin(\pi t/2)$ m/s, state the differential equation with initial condition satisfied by $s(t)$, and find $s(t)$.

76. Beginning at $t = 0$ s with initial velocity 4 m/s, a particle moves in a straight line with acceleration $a(t) = 3t^{1/2}$ m/s^2. Find the distance traveled after 25 s.

77. A car traveling 25 m/s begins to decelerate at a constant rate of 4 m/s^2. After how many seconds does the car come to a stop and how far will the car have traveled during its deceleration before stopping?

78. At time $t = 1$ s, a particle is traveling at 72 m/s and begins to decelerate at the rate $a(t) = -t^{-1/2}$ until it stops. How far does the particle travel during its deceleration before stopping?

79. A 900-kg rocket is released from a space station. As it burns fuel, the rocket's mass decreases and its velocity increases. Let $v(m)$ be the velocity (in meters per second) as a function of mass m. Find the velocity when $m = 729$ kg if $dv/dm = -50m^{-1/2}$. Assume that $v(900) = 0$ m/s.

80. As water flows through a tube of radius $R = 10$ cm, the velocity v of an individual water particle depends only on its distance r from the center of the tube. The particles at the walls of the tube have zero velocity and $dv/dr = -0.06r$. Determine $v(r)$.

81. Verify the linearity properties of the indefinite integral stated in Theorem 4.

Further Insights and Challenges

82. Find constants c_1 and c_2 such that $F(x) = c_1 x \sin x + c_2 \cos x$ is an antiderivative of $f(x) = x \cos x$.

83. Find constants c_1 and c_2 such that $F(x) = c_1 x e^x + c_2 e^x$ is an antiderivative of $f(x) = xe^x$.

84. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Is it true that $y = F(x)G(x)$ is an antiderivative of $y = f(x)g(x)$? Confirm or provide a counterexample.

85. Suppose that $F'(x) = f(x)$.

(a) Show that $y = \frac{1}{2}F(2x)$ is an antiderivative of $y = f(2x)$.

(b) Find the general antiderivative of $y = f(kx)$ for $k \neq 0$.

86. Find an antiderivative for $f(x) = |x|$.

87. Using Theorem 1, prove that if $F'(x) = f(x)$, where f is a polynomial of degree $n - 1$, then F is a polynomial of degree n. Then prove that if g is any function such that $g^{(n)}(x) = 0$, then g is a polynomial of degree at most n.

88. Show that $F(x) = \dfrac{x^{n+1} - 1}{n + 1}$ is an antiderivative of $y = x^n$ for $n \neq -1$. Then use L'Hôpital's Rule to prove that

$$\lim_{n \to -1} F(x) = \ln x$$

In this limit, x is fixed and n is the variable. This result shows that, although the Power Rule breaks down for $n = -1$, the antiderivative of $y = x^{-1}$ is a limit of antiderivatives of $y = x^n$ as $n \to -1$.

5.4 The Fundamental Theorem of Calculus, Part I

The FTC was first stated clearly by Isaac Newton in 1666, although other mathematicians, including Newton's teacher Isaac Barrow, had discovered versions of it earlier.

Having so far introduced both derivatives and integrals, a very reasonable question is why they appear together in this topic called Calculus. The answer is the Fundamental Theorem of Calculus (FTC), which is one of the most important theorems in all of mathematics. This foundational result reveals an unexpected connection between the two main operations of calculus: differentiation and integration. The theorem has two parts. Although they are closely related, we discuss them in separate sections to emphasize the different ways they are used. The first part of the Fundamental Theorem of Calculus will allow us to compute definite integrals without having to take limits of Riemann sums.

REMINDER
*F is called an **antiderivative** of f if $F'(x) = f(x)$. We say also that F is an **indefinite integral** of f, and we use the notation*

$$\int f(x)\,dx = F(x) + C$$

To explain FTC I, recall a result from Example 5 of Section 5.2:

$$\int_4^7 x^2\,dx = \left(\frac{1}{3}\right)7^3 - \left(\frac{1}{3}\right)4^3 = 93$$