Integrals Involving $f(x) = b^x$

← REMINDER

$b = e^{\ln b}, \quad b^x = e^{(\ln b)x}$

The exponential function $f(x) = e^x$ is particularly convenient because e^x is both its own derivative and its own antiderivative. For other bases b, we have

$$\frac{d}{dx}b^x = \frac{d}{dx}e^{(\ln b)x} = (\ln b)e^{(\ln b)x} = (\ln b)b^x \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{b^x}{\ln b}\right) = b^x$$

This translates into the integral formula

$$\int b^x\,dx = \frac{b^x}{\ln b} + C \qquad \boxed{5}$$

■ **EXAMPLE 4** Evaluate $\int_3^5 7^x\,dx$.

Solution Apply Eq. (5) with $b = 7$:

$$\int_3^5 7^x\,dx = \left.\frac{7^x}{\ln 7}\right|_3^5 = \frac{7^5 - 7^3}{\ln 7} \approx 8460.8$$ ■

■ **EXAMPLE 5** Evaluate $\int_0^{\pi/2} (\cos\theta)10^{\sin\theta}\,d\theta$.

Solution Use the substitution $u = \sin\theta$, $du = \cos\theta\,d\theta$. The new limits of integration become $u(0) = 0$ and $u(\pi/2) = 1$:

$$\int_0^{\pi/2} (\cos\theta)10^{\sin\theta}\,d\theta = \int_0^1 10^u\,du = \left.\frac{10^u}{\ln 10}\right|_0^1 = \frac{10^1 - 10^0}{\ln 10} \approx 3.91$$ ■

5.8 SUMMARY

- Integral formula for the natural logarithm:

$$\ln x = \int_1^x \frac{dt}{t}$$

- Integral formulas:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\int \frac{dx}{x^2+1} = \tan^{-1} x + C$$

$$\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + C$$

- Integrals of exponential functions ($b > 0$, $b \neq 1$):

$$\int e^x\,dx = e^x + C, \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

5.8 EXERCISES

Preliminary Questions

1. Find b such that $\int_1^b \frac{dx}{x}$ is equal to

(a) $\ln 3$. **(b)** 3.

2. Find b such that $\int_0^b \frac{dx}{1+x^2} = \frac{\pi}{3}$.

3. Which integral should be evaluated using substitution?

(a) $\int \frac{9\,dx}{1+x^2}$ **(b)** $\int \frac{dx}{1+9x^2}$

4. Which relation between x and u yields $\sqrt{16+x^2} = 4\sqrt{1+u^2}$?

Exercises

In Exercises 1–10, evaluate the definite integral.

1. $\int_1^9 \frac{dx}{x}$

2. $\int_4^{20} \frac{dx}{x}$

3. $\int_1^{e^3} \frac{1}{t}\,dt$

4. $\int_{-e^2}^{-e} \frac{1}{t}\,dt$

5. $\int_2^{12} \frac{dt}{3t+4}$

6. $\int_e^{e^3} \frac{dt}{t\ln t}$

7. $\int_1^{\sqrt{3}} \frac{dx}{x^2+1}$

8. $\int_2^{7} \frac{x\,dx}{x^2+1}$

9. $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

10. $\int_{-2}^{-2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}$

11. Use the substitution $u = x/3$ to prove

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

12. Use the substitution $u = 2x$ to evaluate $\int \frac{dx}{4x^2+1}$.

In Exercises 13–32, calculate the integral.

13. $\int_0^3 \frac{dx}{x^2+3}$

14. $\int_0^4 \frac{dt}{4t^2+9}$

15. $\int \frac{dt}{\sqrt{1-16t^2}}$

16. $\int_{-1/5}^{1/5} \frac{dx}{\sqrt{4-25x^2}}$

17. $\int \frac{dt}{\sqrt{5-3t^2}}$

18. $\int_{1/(2\sqrt{2})}^{1/2} \frac{dx}{x\sqrt{16x^2-1}}$

19. $\int \frac{dx}{x\sqrt{12x^2-3}}$

20. $\int \frac{x\,dx}{x^4+1}$

21. $\int \frac{dx}{x\sqrt{x^4-1}}$

22. $\int_{-1/2}^{0} \frac{(x+1)\,dx}{\sqrt{1-x^2}}$

23. $\int_{-\ln 2}^{0} \frac{e^x\,dx}{1+e^{2x}}$

24. $\int \frac{\ln(\cos^{-1}x)\,dx}{(\cos^{-1}x)\sqrt{1-x^2}}$

25. $\int \frac{\tan^{-1}x\,dx}{1+x^2}$

26. $\int_1^{\sqrt{3}} \frac{dx}{(\tan^{-1}x)(1+x^2)}$

27. $\int_0^1 3^x\,dx$

28. $\int_0^1 3^{-x}\,dx$

29. $\int_0^{\log_4(3)} 4^x\,dx$

30. $\int_0^1 t\,5^{t^2}\,dt$

31. $\int 9^x \sin(9^x)\,dx$

32. $\int \frac{dx}{\sqrt{5^{2x}-1}}$

In Exercises 33–70, evaluate the integral using the methods covered in the text so far.

33. $\int ye^{y^2}\,dy$

34. $\int \frac{dx}{3x+5}$

35. $\int \frac{x\,dx}{\sqrt{4x^2+9}}$

36. $\int (x - x^{-2})^2\,dx$

37. $\int 7^{-x}\,dx$

38. $\int e^{9-12t}\,dt$

39. $\int \sec^2\theta\tan^7\theta\,d\theta$

40. $\int \frac{\cos(\ln t)\,dt}{t}$

41. $\int \frac{t\,dt}{\sqrt{7-t^2}}$

42. $\int 2^x e^{4x}\,dx$

43. $\int \frac{(3x+2)\,dx}{x^2+4}$

44. $\int \tan(4x+1)\,dx$

45. $\int \frac{dx}{\sqrt{9-4x^2}}$

46. $\int e^t\sqrt{e^t+1}\,dt$

47. $\int (e^{-x} - 4x)\,dx$

48. $\int (7 - e^{10x})\,dx$

49. $\int \frac{e^{2x} - e^{4x}}{e^x}\,dx$

50. $\int \frac{dx}{x\sqrt{25x^2-1}}$

51. $\int \frac{(x+5)\,dx}{\sqrt{4-x^2}}$

52. $\int (t+1)\sqrt{t+1}\,dt$

53. $\int e^x\cos(e^x)\,dx$

54. $\int \frac{e^x}{\sqrt{e^x+1}}\,dx$

55. $\int \frac{dx}{\sqrt{9-16x^2}}$

56. $\int \frac{dx}{(4x-1)\ln(8x-2)}$

57. $\int e^x(e^{2x}+1)^3\,dx$

58. $\int \frac{dx}{x(\ln x)^5}$

59. $\int \frac{x^2\,dx}{x^3+2}$

60. $\int \frac{(3x-1)\,dx}{9-2x+3x^2}$

61. $\int \cot x\,dx$

62. $\int \frac{\cos x}{2\sin x + 3}\,dx$

63. $\int \frac{4\ln x + 5}{x}\,dx$

64. $\int (\sec\theta\tan\theta)5^{\sec\theta}\,d\theta$

65. $\int x3^{x^2}\,dx$

66. $\int \frac{\ln(\ln x)}{x\ln x}\,dx$

67. $\int \cot x\ln(\sin x)\,dx$

68. $\int \frac{t\,dt}{\sqrt{1-t^4}}$

69. $\int t^2\sqrt{t-3}\,dt$

70. $\int \cos x\,5^{-2\sin x}\,dx$

71. Use Figure 4 to prove

$$\int_0^x \sqrt{1-t^2}\,dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x$$

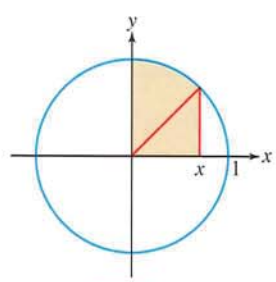

FIGURE 4

72. Use the substitution $u = \tan x$ to evaluate

$$\int \frac{dx}{1+\sin^2 x}$$

Hint: Show that

$$\frac{dx}{1+\sin^2 x} = \frac{du}{1+2u^2}$$

73. Prove

$$\int \sin^{-1} t\, dt = \sqrt{1-t^2} + t\sin^{-1} t$$

74. **(a)** Verify for $r \neq 0$:

$$\int_0^T te^{rt}\, dt = \frac{e^{rT}(rT-1)+1}{r^2} \qquad \boxed{6}$$

Hint: For fixed r, let $F(T)$ be the value of the integral on the left. By FTC II, $F'(t) = te^{rt}$ and $F(0) = 0$. Show that the same is true of the function on the right.
(b) Use L'Hôpital's Rule to show that for fixed T, the limit as $r \to 0$ of the right-hand side of Eq. (6) is equal to the value of the integral for $r = 0$.

Further Insights and Challenges

75. Recall that if $f(t) \geq g(t)$ for $t \geq 0$, then for all $x \geq 0$,

$$\int_0^x f(t)\, dt \geq \int_0^x g(t)\, dt \qquad \boxed{7}$$

The inequality $e^t \geq 1$ holds for $t \geq 0$ because $e > 1$. Use (7) to prove that $e^x \geq 1 + x$ for $x \geq 0$. Then prove, by successive integration, the following inequalities (for $x \geq 0$):

$$e^x \geq 1 + x + \frac{1}{2}x^2, \qquad e^x \geq 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

76. Generalize Exercise 75; that is, use induction (if you are familiar with this method of proof) to prove that for all $n \geq 0$,

$$e^x \geq 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n \quad (x \geq 0)$$

77. Use Exercise 75 to show that $e^x/x^2 \geq x/6$ and conclude that $\lim_{x\to\infty} e^x/x^2 = \infty$. Then use Exercise 76 to prove more generally that $\lim_{x\to\infty} e^x/x^n = \infty$ for all n.

Exercises 78–80 develop an elegant approach to the exponential and logarithm functions. Define a function $G(x)$ for $x > 0$:

$$G(x) = \int_1^x \frac{1}{t}\, dt$$

78. Defining $\ln x$ as an Integral This exercise proceeds as if we didn't know that $G(x) = \ln x$ and shows directly that G has all the basic properties of the logarithm. Prove the following statements:

(a) $\int_a^{ab} \frac{1}{t}\, dt = \int_1^b \frac{1}{t}\, dt$ for all $a, b > 0$. *Hint:* Use the substitution $u = t/a$.
(b) $G(ab) = G(a) + G(b)$. *Hint:* Break up the integral from 1 to ab into two integrals and use (a).
(c) $G(1) = 0$ and $G(a^{-1}) = -G(a)$ for $a > 0$.
(d) $G(a^n) = nG(a)$ for all $a > 0$ and integers n.
(e) $G(a^{1/n}) = \frac{1}{n}G(a)$ for all $a > 0$ and integers $n \neq 0$.
(f) $G(a^r) = rG(a)$ for all $a > 0$ and rational numbers r.
(g) G is increasing. *Hint:* Use FTC II.
(h) There exists a number a such that $G(a) > 1$. *Hint:* Show that $G(2) > 0$ and take $a = 2^m$ for $m > 1/G(2)$.
(i) $\lim_{x\to\infty} G(x) = \infty$ and $\lim_{x\to 0^+} G(x) = -\infty$.
(j) There exists a unique number E such that $G(E) = 1$.
(k) $G(E^r) = r$ for every rational number r.

79. Defining e^x Use Exercise 78 to prove the following statements:
(a) G has an inverse with domain **R** and range $\{x : x > 0\}$. Denote the inverse by F.
(b) $F(x + y) = F(x)F(y)$ for all x, y. *Hint:* It suffices to show that $G(F(x)F(y)) = G(F(x + y))$.
(c) $F(r) = E^r$ for all numbers. In particular, $F(0) = 1$.
(d) $F'(x) = F(x)$. *Hint:* Use the formula for the derivative of an inverse function that appears in Exercise 28 of the Chapter 3 Review Exercises.
This shows that $E = e$ and $F(x) = e^x$ as defined in the text.

80. Defining b^x Let $b > 0$ and let $f(x) = F(xG(b))$ with F as in Exercise 79. Use Exercise 78 (f) to prove that $f(r) = b^r$ for every rational number r. This gives us a way of defining b^x for irrational x, namely $b^x = f(x)$. With this definition, $y = b^x$ is a differentiable function of x (because F is differentiable).

81. The formula $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$ is valid for $n \neq -1$. Show that the exceptional case $n = -1$ is a limit of the general case by applying L'Hôpital's Rule to the limit on the left:

$$\lim_{n\to -1} \int_1^x t^n\, dt = \int_1^x t^{-1}\, dt \qquad \text{(for fixed } x > 0\text{)}$$

Note that the integral on the left is equal to $\frac{x^{n+1}-1}{n+1}$.

82. CAS The integral inside the limit on the left in Exercise 81 is equal to $f_n(x) = \frac{x^{n+1}-1}{n+1}$ for $x \neq -1$. Investigate the limit graphically by plotting $y = f_n(x)$ for $n = 0, -0.3, -0.6$, and -0.9 together with $y = \ln x$ on a single plot.

83. **(a)** Explain why the shaded region in Figure 5 has area $\int_0^{\ln a} e^y\, dy$.
(b) Prove the formula $\int_1^a \ln x\, dx = a\ln a - \int_0^{\ln a} e^y\, dy$.
(c) Conclude that $\int_1^a \ln x\, dx = a\ln a - a + 1$.
(d) Use the result of (a) to find an antiderivative of $y = \ln x$.

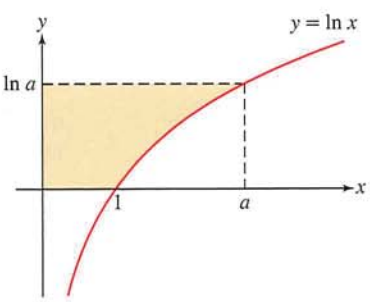

FIGURE 5