For many regions, we have a choice of whether to find the area by integrating with respect to x or with respect to y. The decision is usually based on how easy it is to obtain the curves as functions of one variable in terms of the other, together with how easy it is to subdivide the region into simple regions and then to integrate the functions involved.

EXAMPLE 5 Find the area of the region that is bounded by the three curves $y = x^2$, $y = (x-2)^2$, and $y = 0$.

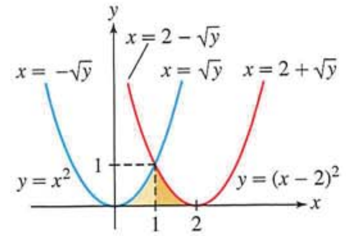


FIGURE 11 This region is horizontally simple but it is easier to cut it into two vertically simple regions.

Solution The area appears in Figure 11. Notice immediately that it is not vertically simple, since the top function changes over the interval $[0, 2]$. It is horizontally simple, but to calculate the area using the fact it is a horizontally simple region will take a bit of work. So first, we do it by splitting the region into two vertically simple regions. Then the area is given by

$$\int_0^1 x^2\,dx + \int_1^2 (x-2)^2\,dx = \left.\frac{x^3}{3}\right|_0^1 + \left.\frac{(x-2)^3}{3}\right|_1^2 = \frac{1}{3} + 0 - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

Now, let's redo the problem, using the fact the region is horizontally simple. We must invert the formulas for the parabolas. The left boundary of the region, which is the right side of the parabola given by $y = x^2$, becomes $x = \sqrt{y}$. To determine the formula for the right boundary of the region, which is the left side of the parabola $y = (x-2)^2$, we solve for x:

$$x - 2 = \pm\sqrt{y}$$
$$x = 2 \pm \sqrt{y}$$

The equation for the left side of the parabola is given by choosing the minus sign, $x = 2 - \sqrt{y}$.

Then the area is given by

$$\int_0^1 \left((2-\sqrt{y}) - \sqrt{y}\right)dy = 2y - \left.\frac{4}{3}y^{3/2}\right|_0^1 = \frac{2}{3}$$

6.1 SUMMARY

- If $f(x) \ge g(x)$ on $[a, b]$, then the area between the graphs is vertically simple and we have

$$\text{Area between the graphs} = \int_a^b \left(y_{\text{top}} - y_{\text{bot}}\right)dx = \int_a^b \left(f(x) - g(x)\right)dx$$

- To calculate the area between $y = f(x)$ and $y = g(x)$, sketch the region to find y_{top}. If necessary, find points of intersection by solving $f(x) = g(x)$.
- Integral along the y-axis: $\int_c^d g(y)\,dy$ is equal to the signed area between the graph and the y-axis for $c \le y \le d$. Area to the right of the y-axis is positive and the area to the left is negative.
- If $g(y) \ge h(y)$ on $[c, d]$, then $x = g(y)$ lies to the right of $x = h(y)$ and the region is horizontally simple.

$$\text{Area between the graphs} = \int_c^d \left(x_{\text{right}} - x_{\text{left}}\right)dy = \int_c^d \left(g(y) - h(y)\right)dy$$

6.1 EXERCISES

Preliminary Questions

1. What is the area interpretation of $\int_a^b \left(f(x) - g(x)\right)dx$ if $f(x) \ge g(x)$?

2. Is $\int_a^b \left(f(x) - g(x)\right)dx$ still equal to the area between the graphs of f and g if $f(x) \ge 0$ but $g(x) \le 0$?

3. Suppose that $f(x) \geq g(x)$ on $[0, 3]$ and $g(x) \geq f(x)$ on $[3, 5]$. Express the area between the graphs over $[0, 5]$ as a sum of integrals.

4. Suppose that the graph of $x = f(y)$ lies to the left of the y-axis. Is $\int_a^b f(y)\,dy$ positive or negative?

5. Explain what $\int_a^b |f(x) - g(x)|\,dx$ represents.

6. Draw a region that is both vertically simple and horizontally simple.

Exercises

1. Find the area of the region between $y = 3x^2 + 12$ and $y = 4x + 4$ over $[-3, 3]$ (Figure 12).

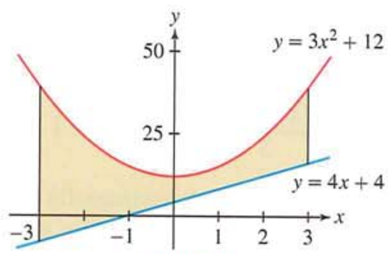


FIGURE 12

2. Find the area of the region between the graphs of $f(x) = 3x + 8$ and $g(x) = x^2 + 2x + 2$ over $[0, 2]$.

3. Find the area of the region enclosed by the graphs of $f(x) = x^2 + 2$ and $g(x) = 2x + 5$ (Figure 13).

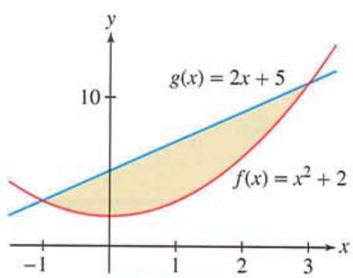


FIGURE 13

4. Find the area of the region enclosed by the graphs of $f(x) = x^3 - 10x$ and $g(x) = 6x$ (Figure 14).

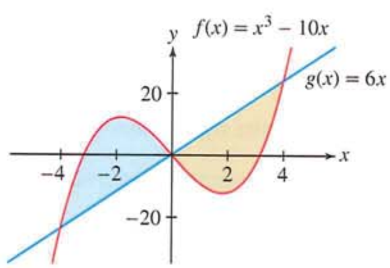


FIGURE 14

In Exercises 5 and 6, sketch the region between $y = \sin x$ and $y = \cos x$ over the interval and find its area.

5. $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

6. $[0, \pi]$

In Exercises 7 and 8, let $f(x) = 20 + x - x^2$ and $g(x) = x^2 - 5x$.

7. Sketch the region enclosed by the graphs of f and g, and compute its area.

8. Sketch the region between the graphs of f and g over $[4, 8]$, and compute its area as a sum of two integrals.

9. Find the area between $y = e^x$ and $y = e^{2x}$ over $[0, 1]$.

10. Find the area of the region bounded by $y = e^x$ and $y = 12 - e^x$ and the y-axis.

11. Sketch the region bounded by the line $y = 2$ and the graph of $y = \sec^2 x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and find its area.

12. Sketch the region bounded by

$$y = \sqrt{4 - x^2} \quad \text{and} \quad y = -\sqrt{4 - x^2}$$

for $-2 \leq x \leq 2$. Write down a definite integral that gives its area, but then use geometry to find its area (and thereby determine the integral).

In Exercises 13–16, determine whether or not the region bounded by the curves is vertically simple and/or horizontally simple.

13. $x = y^2$, $x = 2 - y^2$

14. $y = x^2$, $x = y^2$

15. $y = x$, $y = 2x$, $y = \frac{1}{x}$

16. $y = \sin x$ for $0 \leq x \leq \pi$ and $y = 1 - x$, $y = 0$ (Of the two regions that are bounded by these curves, take the one that does not contain the origin on its boundary.)

In Exercises 17–20, find the area of the shaded region in Figures 15–18.

17.

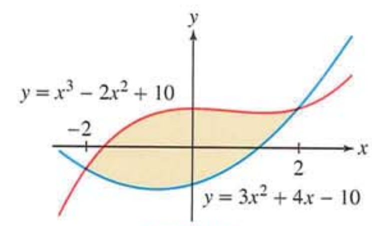


FIGURE 15

18.

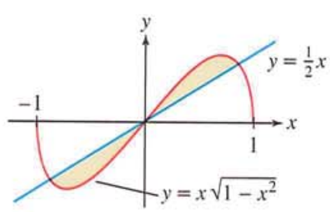


FIGURE 16

19.

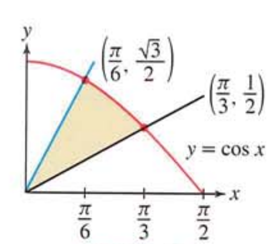


FIGURE 17

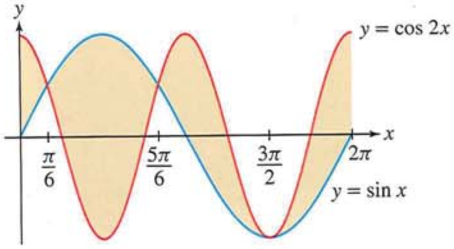
20. 

FIGURE 18

In Exercises 21 and 22, find the area between the graphs of $x = \sin y$ and $x = 1 - \cos y$ over the given interval (Figure 19).

21. $0 \le y \le \frac{\pi}{2}$

22. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

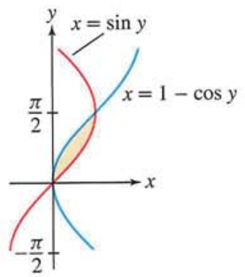


FIGURE 19

23. Find the area of the region lying to the right of $x = y^2 + 4y - 22$ and to the left of $x = 3y + 8$.

24. Find the area of the region lying to the right of $x = y^2 - 5$ and to the left of $x = 3 - y^2$.

25. Figure 20 shows the region enclosed by $x = y^3 - 26y + 10$ and $x = 40 - 6y^2 - y^3$. Match the equations with the curves and compute the area of the region.

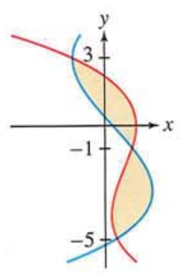


FIGURE 20

26. Figure 21 shows the region enclosed by $y = x^3 - 6x$ and $y = 8 - 3x^2$. Match the equations with the curves and compute the area of the region.

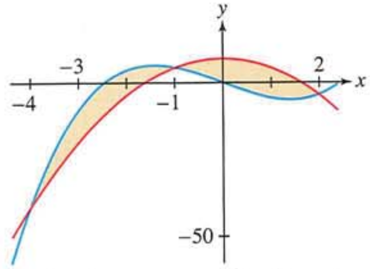


FIGURE 21 Region between $y = x^3 - 6x$ and $y = 8 - 3x^2$.

In Exercises 27 and 28, find the area enclosed by the graphs in two ways: by integrating along the x-axis and by integrating along the y-axis.

27. $x = 9 - y^2, \quad x = 5$

28. The *semicubical parabola* $y^2 = x^3$ and the line $x = 1$

In Exercises 29 and 30, find the area of the region using the method (integration along either the x- or the y-axis) that requires you to evaluate just one integral.

29. Region between $y^2 = x + 5$ and $y^2 = 3 - x$

30. Region between $y = x$ and $x + y = 8$ over $[2, 3]$

In Exercises 31–48, sketch the region enclosed by the curves and compute its area as an integral along the x- or y-axis.

31. $y = 4 - x^2, \quad y = x^2 - 4$

32. $y = x^2 - 6, \quad y = 6 - x^3, \quad$ y-axis

33. $x + y = 4, \quad x - y = 0, \quad y + 3x = 4$

34. $y = 8 - 3x, \quad y = 6 - x, \quad y = 2$

35. $y = 8 - \sqrt{x}, \quad y = \sqrt{x}, \quad x = 0$

36. $y = \frac{x}{x^2 + 1}, \quad y = \frac{x}{5}$

37. $x = |y|, \quad x = 1 - |y|$

38. $y = |x|, \quad y = x^2 - 6$

39. $x = y^3 - 18y, \quad y + 2x = 0$

40. $y = x\sqrt{x - 2}, \quad y = -x\sqrt{x - 2}, \quad x = 4$

41. $x = 2y, \quad x + 1 = (y - 1)^2$

42. $x + y = 1, \quad x^{1/2} + y^{1/2} = 1$

43. $y = \cos x, \quad y = \cos 2x, \quad x = 0, \quad x = \frac{2\pi}{3}$

44. $y = \tan x, \quad y = -\tan x, \quad x = \frac{\pi}{4}$

45. $y = \sin x, \quad y = \csc^2 x, \quad x = \frac{\pi}{4}$

46. $x = \sin y, \quad x = \frac{2}{\pi} y$

47. $y = e^x, \quad y = e^{-x}, \quad y = 2$

48. $y = \frac{\ln x}{x}, \quad y = \frac{(\ln x)^2}{x}$

49. CAS Plot

$$y = \frac{x}{\sqrt{x^2 + 1}} \quad \text{and} \quad y = (x - 1)^2$$

on the same set of axes. Use a computer algebra system to find the points of intersection numerically and compute the area between the curves.

50. Sketch a region whose area is represented by

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\sqrt{1 - x^2} - |x|\right) dx$$

and evaluate using geometry.

51. Athletes 1 and 2 run along a straight track with velocities $v_1(t)$ and $v_2(t)$ (in meters per second) as shown in Figure 22.

(a) Which of the following is represented by the area of the shaded region over [0, 10]?

i. The distance between athletes 1 and 2 at time $t = 10$ s

ii. The difference in the distance traveled by the athletes over the time interval [0, 10]

(b) Does Figure 22 give us enough information to determine who is ahead at time $t = 10$ s?

(c) If the athletes begin at the same time and place, who is ahead at $t = 10$ s? At $t = 25$ s?

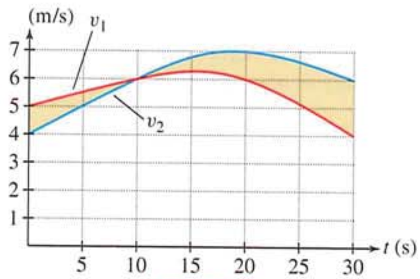


FIGURE 22

52. Express the area (not signed) of the shaded region in Figure 23 as a sum of three integrals involving $f(x)$ and $g(x)$.

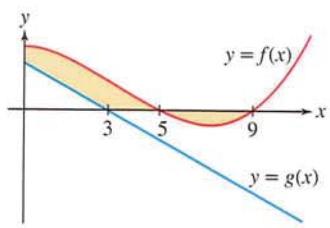


FIGURE 23

53. Find the area enclosed by the curves $y = c - x^2$ and $y = x^2 - c$ as a function of c. Find the value of c for which this area is equal to 1.

54. Set up (but do not evaluate) an integral that expresses the area between the circles $x^2 + y^2 = 2$ and $x^2 + (y-1)^2 = 1$.

55. Set up (but do not evaluate) an integral that expresses the area between the graphs of $y = (1 + x^2)^{-1}$ and $y = x^2$.

56. CAS Find a numerical approximation to the area above $y = 1 - (x/\pi)$ and below $y = \sin x$ (find the points of intersection numerically).

57. CAS Find a numerical approximation to the area above $y = |x|$ and below $y = \cos x$.

58. CAS Use a computer algebra system to find a numerical approximation to the number c (besides zero) in $\left[0, \frac{\pi}{2}\right]$, where the curves $y = \sin x$ and $y = \tan^2 x$ intersect. Then find the area enclosed by the graphs over $[0, c]$.

59. The back of Jon's guitar (Figure 24) is 19 inches (in.) long. Jon measured the width at 1-in. intervals, beginning and ending $\frac{1}{2}$ in. from the ends, obtaining the results

6, 9, 10.25, 10.75, 10.75, 10.25, 9.75, 9.5, 10, 11.25,

12.75, 13.75, 14.25, 14.5, 14.5, 14, 13.25, 11.25, 9

Use the midpoint rule to estimate the area of the back.

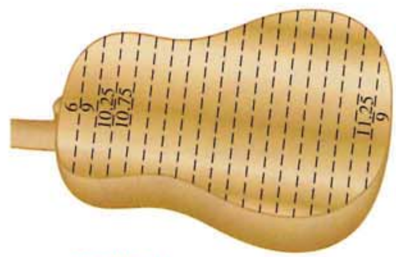


FIGURE 24 Back of guitar.

60. Referring to Figure 1 at the beginning of this section, estimate the projected number of additional joules produced in the years 2009–2030 as a result of government stimulus spending in 2009–2010. *Note:* One watt (W) is equal to 1 joule/second (J/s), and 1 gigawatt (GW) is 10^9 watts.

Exercises 61 and 62 use the notation and results of Exercises 49–51 of Section 3.4. For a given country, $F(r)$ is the fraction of total income that goes to the bottom rth fraction of households. The graph of $y = F(r)$ is called the Lorenz curve.

61. Let A be the area between $y = r$ and $y = F(r)$ over the interval [0, 1] (Figure 25). The **Gini index** is the ratio $G = A/B$, where B is the area under $y = r$ over [0, 1].

(a) Show that

$$G = 2\int_0^1 (r - F(r))\,dr$$

(b) Calculate G if

$$F(r) = \begin{cases} \frac{1}{3}r & \text{for } 0 \le r \le \frac{1}{2} \\ \frac{5}{3}r - \frac{2}{3} & \text{for } \frac{1}{2} \le r \le 1 \end{cases}$$

(c) The Gini index is a measure of income distribution, with a lower value indicating a more equal distribution. Calculate G if $F(r) = r$ (in this case, all households have the same income by Exercise 51(b) of Section 3.4).

(d) What is G if all of the income goes to one household? *Hint:* In this extreme case, $F(r) = 0$ for $0 \le r < 1$.

62. Calculate the Gini index of the United States in the year 2010 from the Lorenz curve in Figure 25, which consists of segments joining the data points in the following table:

r	0	0.2	0.4	0.6	0.8	1
$F(r)$	0	0.033	0.118	0.264	0.480	1

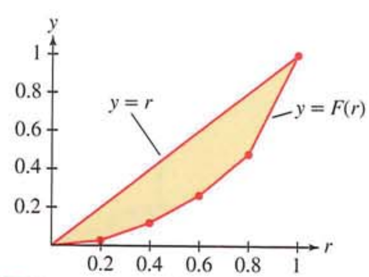


FIGURE 25 Lorenz curve for the United States in 2010.