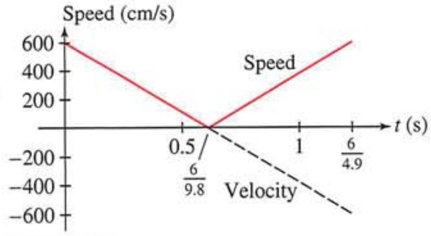


FIGURE 15 Graph of speed $|h'(t)| = |600 - 980t|$.

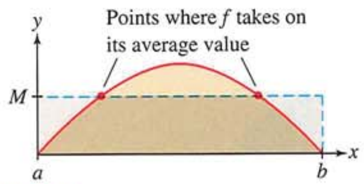


FIGURE 16 The function f takes on its average value M at the points where the upper edge of the rectangle intersects the graph.

The average speed $\bar{s}$ is

$$\bar{s} = \frac{1}{\frac{6}{4.9}} \int_0^{6/4.9} |600 - 980t| \, dt = \frac{1}{\frac{6}{4.9}} \left(\frac{3600}{9.8} \right) = 300 \text{ cm/s}$$

There is an important difference between the average of a list of numbers and the average value of a continuous function. If the average score on an exam is 84, then 84 lies between the highest and lowest scores, but it is possible that no student received a score of 84. By contrast, the Mean Value Theorem (MVT) for Integrals asserts that a continuous function always takes on its average value somewhere in the interval (Figure 16).

For example, the average of $f(x) = \sin x$ on $[0, \pi]$ is $2/\pi$ by Example 7. We have $f(c) = 2/\pi$ for $c = \sin^{-1}(2/\pi) \approx 0.69$. Since 0.69 lies in $[0, \pi]$, $f(x) = \sin x$ indeed takes on its average value at a point in the interval.

THEOREM 1 Mean Value Theorem for Integrals If f is continuous on $[a, b]$, then there exists a value $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx$$

Proof Let $M = \dfrac{1}{b-a} \displaystyle\int_a^b f(x) \, dx$ be the average value. Because f is continuous, we can apply Theorem 1 of Section 4.2 to conclude that f takes on a minimum value m_{min} and a maximum value M_{max} on the closed interval $[a, b]$. Furthermore, by Eq. (8) of Section 5.2,

$$m_{\text{min}}(b-a) \le \int_a^b f(x) \, dx \le M_{\text{max}}(b-a)$$

Dividing by $(b - a)$, we find

$$m_{\text{min}} \le M \le M_{\text{max}}$$

In other words, the average value M lies between m_{min} and M_{max}. The Intermediate Value Theorem guarantees that $f(x)$ takes on every value between its min and max, so $f(c) = M$ for some c in $[a, b]$.

6.2 SUMMARY

- Formulas:

Volume	$V = \int_a^b A(y) \, dy$,	$A(y) =$ cross-sectional area
Total Mass	$M = \int_a^b \rho(x) \, dx$,	$\rho(x) =$ linear mass density
Total Population	$P = 2\pi \int_0^R r\rho(r) \, dr$,	$\rho(r) =$ radial density
Laminar Flow Rate	$Q = 2\pi \int_0^R rv(r) \, dr$,	$v(r) =$ velocity at radius r
Average value	$M = \frac{1}{b-a} \int_a^b f(x) \, dx$,	$f =$ any continuous function

- The MVT for Integrals: If f is continuous on $[a, b]$ with average (or mean) value M, then $f(c) = M$ for some $c \in [a, b]$.

6.2 EXERCISES

Preliminary Questions

1. What is the average value of f on $[0, 4]$ if the area between the graph of f and the x-axis is equal to 12?

2. Find the volume of a solid extending from $y = 2$ to $y = 5$ if every cross section has area $A(y) = 5$.

3. What is the definition of flow rate?

4. Which assumption about fluid velocity did we use to compute the flow rate as an integral?

5. The average value of f on $[1, 4]$ is 5. Find $\int_1^4 f(x)\,dx$.

Exercises

1. Let V be the volume of a pyramid of height 20 whose base is a square of side 8.

(a) Use similar triangles as in Example 1 to find the area of the horizontal cross section at a height y.

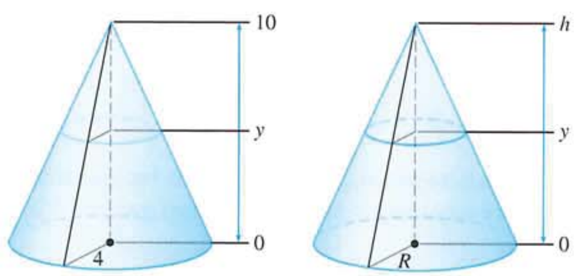
(b) Calculate V by integrating the cross-sectional area.

2. Let V be the volume of a right circular cone of height 10 whose base is a circle of radius 4 [Figure 17(A)].

(a) Use similar triangles to find the area of a horizontal cross section at a height y.

(b) Calculate V by integrating the cross-sectional area.

3. Use the method of Exercise 2 to find the formula for the volume of a right circular cone of height h whose base is a circle of radius R [Figure 17(B)].



(A) (B)

FIGURE 17 Right circular cones.

4. Calculate the volume of the ramp in Figure 18 in three ways by integrating the area of the cross sections:

(a) Perpendicular to the x-axis (rectangles)

(b) Perpendicular to the y-axis (triangles)

(c) Perpendicular to the z-axis (rectangles)

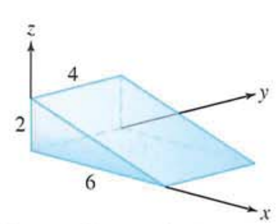


FIGURE 18 Ramp of length 6, width 4, and height 2.

5. Find the volume of liquid needed to fill a sphere of radius R to height h (Figure 19).

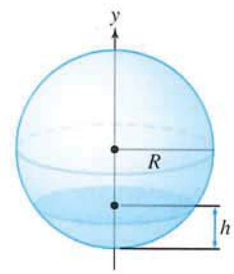
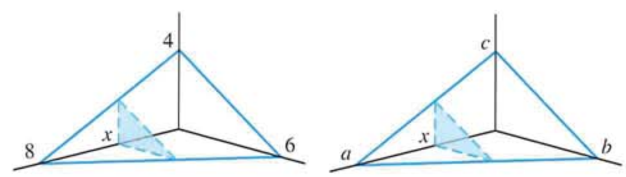


FIGURE 19 Sphere filled with liquid to height h.

6. Find the volume of the wedge in Figure 20(A) by integrating the area of vertical cross sections.



(A) (B)

FIGURE 20

7. Derive a formula for the volume of the wedge in Figure 20(B) in terms of the constants a, b, and c.

8. Let B be the solid whose base is the unit circle $x^2 + y^2 = 1$ and whose vertical cross sections perpendicular to the x-axis are equilateral triangles. Show that the vertical cross sections have area $A(x) = \sqrt{3}(1 - x^2)$ and compute the volume of B.

In Exercises 9–14, find the volume of the solid with the given base and cross sections.

9. The base is the unit circle $x^2 + y^2 = 1$, and the cross sections perpendicular to the x-axis are triangles whose height and base are equal.

10. The base is the triangle enclosed by $x + y = 1$, the x-axis, and the y-axis. The cross sections perpendicular to the y-axis are semicircles.

11. The base is the semicircle $y = \sqrt{9 - x^2}$, where $-3 \le x \le 3$. The cross sections perpendicular to the x-axis are squares.

12. The base is a square, one of whose sides is the interval $[0, \ell]$ along the x-axis. The cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.

13. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are squares.

14. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the y-axis are rectangles of height y^3.

15. Find the volume of the solid whose base is the region $|x| + |y| \le 1$ and whose vertical cross sections perpendicular to the y-axis are semicircles (with diameter along the base).

16. Show that a pyramid of height h whose base is an equilateral triangle of side s has volume $\frac{\sqrt{3}}{12}hs^2$.

17. The area of an ellipse is πab, where a and b are the lengths of the semimajor and semiminor axes (Figure 21). Compute the volume of a cone of height 12 whose base is an ellipse with semimajor axis $a = 6$ and semiminor axis $b = 4$.

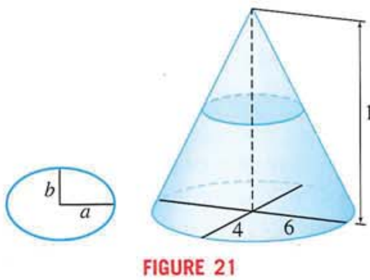


FIGURE 21

18. Find the volume V of a *regular* tetrahedron (Figure 22) whose face is an equilateral triangle of side s. The tetrahedron has height $h = \sqrt{2/3}s$.

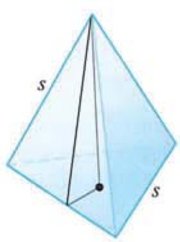


FIGURE 22 Regular tetrahedron.

19. A frustum of a pyramid is a pyramid with its top cut off [Figure 23(A)]. Let V be the volume of a frustum of height h whose base is a square of side a and whose top is a square of side b with $a > b \ge 0$.

(a) Show that if the frustum were continued to a full pyramid, it would have height $ha/(a-b)$ [Figure 23(B)].

(b) Show that the cross section at height x is a square of side $(1/h)(a(h-x)+bx)$.

(c) Show that $V = \frac{1}{3}h(a^2 + ab + b^2)$. A papyrus dating to the year 1850 BCE indicates that Egyptian mathematicians had discovered this formula almost 4000 years ago.

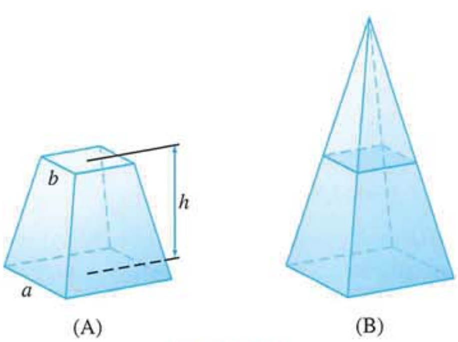


FIGURE 23

20. A plane inclined at an angle of 45° passes through a diameter of the base of a cylinder of radius r. Find the volume of the region within the cylinder and below the plane (Figure 24).

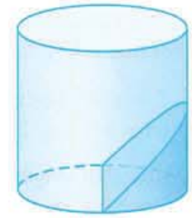


FIGURE 24

21. The solid S in Figure 25 is the intersection of two cylinders of radius r whose axes are perpendicular.

(a) The horizontal cross section of each cylinder at distance y from the central axis is a rectangular strip. Find the strip's width.

(b) Find the area of the horizontal cross section of S at distance y.

(c) Find the volume of S as a function of r.

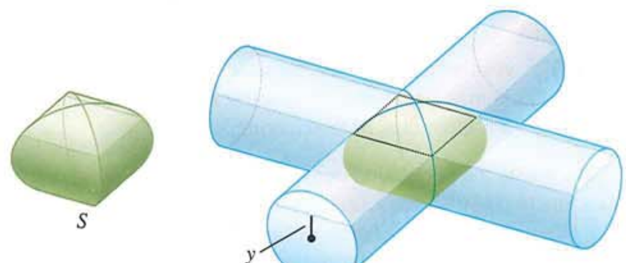


FIGURE 25 Two cylinders intersecting at right angles.

22. Let S be the intersection of two cylinders of radius r whose axes intersect at an angle θ. Find the volume of S as a function of r and θ.

23. Calculate the volume of a cylinder inclined at an angle $\theta = 30°$ with height 10 and base of radius 4 (Figure 26).

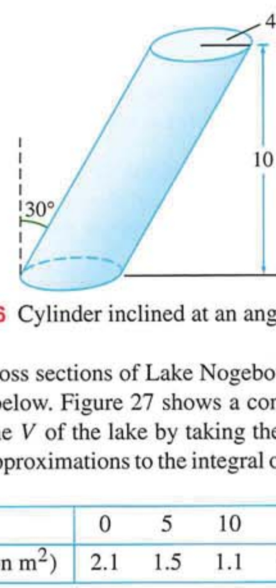


FIGURE 26 Cylinder inclined at an angle $\theta = 30°$.

24. The areas of cross sections of Lake Nogebow at 5-m intervals are given in the table below. Figure 27 shows a contour map of the lake. Estimate the volume V of the lake by taking the average of the right- and left-endpoint approximations to the integral of cross-sectional area.

Depth (m)	0	5	10	15	20
Area (million m^2)	2.1	1.5	1.1	0.835	0.217

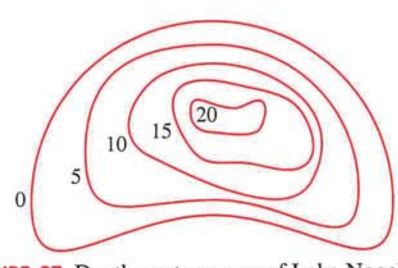


FIGURE 27 Depth contour map of Lake Nogebow.

25. Find the total mass of a 1-m rod whose linear density function is $\rho(x) = 10(x+1)^{-2}$ kg/m for $0 \le x \le 1$.

26. Find the total mass of a 2-m rod whose linear density function is $\rho(x) = 1 + 0.5\sin(\pi x)$ kg/m for $0 \le x \le 2$.

27. A mineral deposit along a strip of length 6 cm has density $s(x) = 0.01x(6-x)$ g/cm for $0 \le x \le 6$. Calculate the total mass of the deposit.

28. Charge is distributed along a glass tube of length 10 cm with linear charge density $\rho(x) = x(x^2+1)^{-2} \times 10^{-4}$ coulombs per centimeter (C/cm) for $0 \le x \le 10$. Calculate the total charge.

29. Calculate the population within a 10-mile radius of the city center if the radial population density is $\rho(r) = 4(1+r^2)^{1/3}$ (in thousands per square mile).

30. Odzala National Park in the Republic of the Congo has a high density of gorillas. Suppose that the radial population density is $\rho(r) = 52(1+r^2)^{-2}$ gorillas per square kilometer, where r is the distance from a grassy clearing with a source of water. Calculate the number of gorillas within a 5-km radius of the clearing.

31. Table 1 lists the population density (in people per square kilometer) as a function of distance r (in kilometers) from the center of a rural town. Estimate the total population within a 1.2-km radius of the center by taking the average of the left- and right-endpoint approximations.

TABLE 1 Population Density

r	$\rho(r)$	r	$\rho(r)$
0.0	125.0	0.8	56.2
0.2	102.3	1.0	46.0
0.4	83.8	1.2	37.6
0.6	68.6		

32. Find the total mass of a circular plate of radius 20 cm whose mass density is the radial function $\rho(r) = 0.03 + 0.01\cos(\pi r^2)$ g/cm^2.

33. The density of deer in a forest is the radial function $\rho(r) = 150(r^2+2)^{-2}$ deer per square kilometer, where r is the distance (in kilometers) to a small meadow. Calculate the number of deer in the region $2 \le r \le 5$ km.

34. Show that a circular plate of radius 2 cm with radial mass density $\rho(r) = \frac{4}{r}$ g/cm^2 has finite total mass, even though the density becomes infinite at the origin.

35. Find the flow rate through a tube of radius 4 cm, assuming that the velocity of fluid particles at a distance r centimeters from the center is $v(r) = (16 - r^2)$ cm/s.

36. The velocity of fluid particles flowing through a tube of radius 5 cm is $v(r) = (10 - 0.3r - 0.34r^2)$ cm/s, where r centimeters is the distance from the center. What quantity per second of fluid flows through the portion of the tube where $0 \le r \le 2$?

37. A solid rod of radius 1 cm is placed in a pipe of radius 3 cm so that their axes are aligned. Water flows through the pipe and around the rod. Find the flow rate if the velocity of the water is given by the radial function $v(r) = 0.5(r-1)(3-r)$ cm/s.

38. Let $v(r)$ be the velocity of blood in an arterial capillary of radius $R = 4 \times 10^{-5}$ m. Use Poiseuille's Law (Example 6) with $k = 10^6$ (m-s)$^{-1}$ to determine the velocity at the center of the capillary and the flow rate (use correct units).

In Exercises 39–48, calculate the average over the given interval.

39. $f(x) = x^3$, $[0, 4]$

40. $f(x) = x^3$, $[-1, 1]$

41. $f(x) = \cos x$, $\left[0, \frac{\pi}{6}\right]$

42. $f(x) = \sec^2 x$, $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

43. $f(s) = s^{-2}$, $[2, 5]$

44. $f(x) = \dfrac{\sin(\pi/x)}{x^2}$, $[1, 2]$

45. $f(x) = 2x^3 - 6x^2$, $[-1, 3]$

46. $f(x) = \dfrac{1}{x^2+1}$, $[-1, 1]$

47. $f(x) = x^n$ for $n \ge 0$, $[0, 1]$

48. $f(x) = e^{-nx}$, $[-1, 1]$

49. The temperature (in degrees Celsius) at time t (in hours) in an art museum varies according to $T(t) = 20 + 5\cos\left(\frac{\pi}{12}t\right)$. Find the average over the time periods $[0, 24]$ and $[2, 6]$.

50. A steel bar of length 3 m experiences extreme heat at its center, so that the temperature at coordinate x on the bar is given by $T(x) = 40\sin\left(\frac{\pi x}{3}\right) + 50$°C where the bar sits along the interval $[0, 3]$ on the x-axis. Determine the average temperature of the bar.

51. Temperature in the town of Walla Walla during the month of July follows a pattern given by $T(t) = 10\sin\left(\frac{t\pi}{31}\right) + 14\sin\left(\frac{t\pi}{2}\right) + 73$°F. Here, t is measured in days, and there are 31 days in July. Explain why you might see a pattern like this and compute the average temperature during the month of July.

52. The door to the garage is left open and over the next 4 hours (h), the temperature in a house in degrees Celsius is given by $T(t) = 20e^{-t/4}$. Determine the average temperature over those 4 h.

53. A 10-cm copper wire with one end in an ice bath is heated at the other end, so that the temperature at each point x along the wire (in degrees Celsius) is given by $T(x) = 50\cos\frac{\pi x}{20}$. Find the average temperature over the wire.

54. A ball thrown in the air vertically from ground level with initial velocity 18 m/s has height $h(t) = 18t - 9.8t^2$ at time t (in seconds). Find the average height and the average speed over the time interval extending from the ball's release to its return to ground level.

55. Find the average speed over the time interval $[1, 5]$ (time in seconds) of a particle whose position at time t is $s(t) = t^3 - 6t^2$ m.

56. An object with zero initial velocity accelerates at a constant rate of 10 m/s^2. Find its average velocity during the first 15 s.

57. The acceleration of a particle is $a(t) = 60t - 4t^3$ m/s^2. Compute the average acceleration and the average speed over the time interval $[2, 6]$, assuming that the particle's initial velocity is zero.

58. What is the average area of the circles whose radii vary from 0 to R?

59. Let M be the average value of $f(x) = x^4$ on $[0, 3]$. Find a value of c in $[0, 3]$ such that $f(c) = M$.

60. Let $f(x) = \sqrt{x}$. Find a value of c in $[4, 9]$ such that $f(c)$ is equal to the average of f on $[4, 9]$.

61. Let M be the average value of $f(x) = x^3$ on $[0, A]$, where $A > 0$. Which theorem guarantees that $f(c) = M$ has a solution c in $[0, A]$? Find c.

62. CAS Let $f(x) = 2\sin x - x$. Use a computer algebra system to plot f and estimate:

(a) the positive root α of f.

(b) the average value M of f on $[0, \alpha]$.

(c) a value $c \in [0, \alpha]$ such that $f(c) = M$.