Exercises

1. Which of the following integrals is improper? Explain your answer, but do not evaluate the integral.

(a) $\displaystyle\int_0^2 \frac{dx}{x^{1/3}}$ **(b)** $\displaystyle\int_1^\infty \frac{dx}{x^{0.2}}$ **(c)** $\displaystyle\int_{-1}^\infty e^{-x}\,dx$

(d) $\displaystyle\int_0^1 e^{-x}\,dx$ **(e)** $\displaystyle\int_0^\pi \sec x\,dx$ **(f)** $\displaystyle\int_0^\infty \sin x\,dx$

(g) $\displaystyle\int_0^1 \sin x\,dx$ **(h)** $\displaystyle\int_0^1 \frac{dx}{\sqrt{3-x^2}}$ **(i)** $\displaystyle\int_1^\infty \ln x\,dx$

(j) $\displaystyle\int_0^3 \ln x\,dx$

2. Let $f(x) = x^{-4/3}$.

(a) Evaluate $\displaystyle\int_1^R f(x)\,dx$.

(b) Evaluate $\displaystyle\int_1^\infty f(x)\,dx$ by computing the limit

$$\lim_{R\to\infty}\int_1^R f(x)\,dx$$

3. Prove that $\displaystyle\int_1^\infty x^{-2/3}\,dx$ diverges by showing that

$$\lim_{R\to\infty}\int_1^R x^{-2/3}\,dx = \infty$$

4. Determine whether $\displaystyle\int_0^3 \frac{dx}{(3-x)^{3/2}}$ converges by computing

$$\lim_{R\to 3-}\int_0^R \frac{dx}{(3-x)^{3/2}}$$

In Exercises 5–40, determine whether the improper integral converges and, if so, evaluate it.

5. $\displaystyle\int_1^\infty \frac{dx}{x^{19/20}}$ **6.** $\displaystyle\int_1^\infty \frac{dx}{x^{20/19}}$

7. $\displaystyle\int_{-\infty}^4 e^{0.0001t}\,dt$ **8.** $\displaystyle\int_{20}^\infty \frac{dt}{t}$

9. $\displaystyle\int_0^5 \frac{dx}{x^{20/19}}$ **10.** $\displaystyle\int_0^5 \frac{dx}{x^{19/20}}$

11. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$ **12.** $\displaystyle\int_5^6 \frac{dx}{(x-5)^{3/2}}$

13. $\displaystyle\int_2^\infty x^{-3}\,dx$ **14.** $\displaystyle\int_0^\infty \frac{dx}{(x+1)^3}$

15. $\displaystyle\int_{-3}^\infty \frac{dx}{(x+4)^{3/2}}$ **16.** $\displaystyle\int_2^\infty e^{-2x}\,dx$

17. $\displaystyle\int_{-1}^1 \frac{dx}{x^{0.2}}$ **18.** $\displaystyle\int_2^\infty x^{-1/3}\,dx$

19. $\displaystyle\int_4^\infty e^{-3x}\,dx$ **20.** $\displaystyle\int_4^\infty e^{3x}\,dx$

21. $\displaystyle\int_{-\infty}^0 e^{3x}\,dx$ **22.** $\displaystyle\int_1^2 \frac{dx}{(x-1)^2}$

23. $\displaystyle\int_1^3 \frac{dx}{\sqrt{3-x}}$ **24.** $\displaystyle\int_{-4}^0 \frac{dx}{(x+2)^{1/3}}$

25. $\displaystyle\int_0^\infty \frac{dx}{1+x}$ **26.** $\displaystyle\int_{-\infty}^0 xe^{-x^2}\,dx$

27. $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x^2)^2}$ **28.** $\displaystyle\int_3^6 \frac{x\,dx}{\sqrt{x-3}}$

29. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx$ **30.** $\displaystyle\int_1^\infty xe^{-2x}\,dx$

31. $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ **32.** $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$

33. $\displaystyle\int_1^\infty \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$ **34.** $\displaystyle\int_0^\pi \sec\theta\,d\theta$

35. $\displaystyle\int_0^\infty \sin x\,dx$ **36.** $\displaystyle\int_0^{\pi/2} \tan x\,dx$

37. $\displaystyle\int_0^1 \ln x\,dx$ **38.** $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

39. $\displaystyle\int_0^1 \frac{\ln x}{x^2}\,dx$ **40.** $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

41. Let $I = \displaystyle\int_4^\infty \frac{dx}{(x-2)(x-3)}$.

(a) Show that for $R > 4$,

$$\int_4^R \frac{dx}{(x-2)(x-3)} = \ln\left|\frac{R-3}{R-2}\right| - \ln\frac{1}{2}$$

(b) Then show that $I = \ln 2$.

42. Evaluate the integral $I = \displaystyle\int_1^\infty \frac{dx}{x(2x+5)}$.

43. Evaluate $I = \displaystyle\int_0^1 \frac{dx}{x(2x+5)}$ or state that it diverges.

44. Evaluate $I = \displaystyle\int_2^\infty \frac{dx}{(x+3)(x+1)^2}$ or state that it diverges.

In Exercises 45–48, determine whether the doubly infinite improper integral converges and, if so, evaluate it. Use definition (2).

45. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{1+x^2}$ **46.** $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx$

47. $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\,dx$ **48.** $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)^{3/2}}$

49. Determine whether $J = \displaystyle\int_{-1}^1 \frac{dx}{x^{1/3}}$ converges and, if so, to what.

50. Consider the integral $\displaystyle\int_{-\infty}^\infty x\,dx$.

(a) Show that it diverges.

(b) Show that $\displaystyle\lim_{R\to\infty}\int_{-R}^R x\,dx$ converges, thereby demonstrating that the definition of $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ needs to be adhered to carefully.

51. For which values of a does $\displaystyle\int_0^\infty e^{ax}\,dx$ converge?

52. Show that $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges if $p < 1$ and diverges if $p \geq 1$.

53. Sketch the region under the graph of $f(x) = \dfrac{1}{1+x^2}$ for $-\infty < x < \infty$, and show that its area is π.

54. Show that $\dfrac{1}{\sqrt{x^4+1}} \le \dfrac{1}{x^2}$ for all x, and use this to prove that $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^4+1}}$ converges.

55. Show that $\displaystyle\int_1^\infty \frac{dx}{x^3+4}$ converges by comparing with $\displaystyle\int_1^\infty x^{-3}\,dx$.

56. Show that $\displaystyle\int_2^\infty \frac{dx}{x^3-4}$ converges by comparing with $\displaystyle\int_2^\infty 2x^{-3}\,dx$.

57. Show that $0 \le e^{-x^2} \le e^{-x}$ for $x \ge 1$ (Figure 11). Use the Comparison Test to show that $\int_0^\infty e^{-x^2}\,dx$ converges. *Hint:* It suffices (why?) to make the comparison for $x \ge 1$ because

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

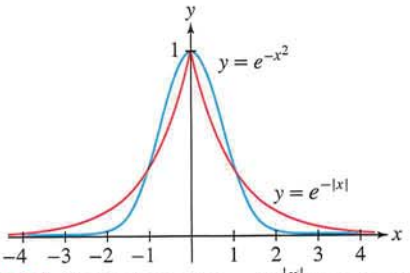

FIGURE 11 Comparison of $y = e^{-|x|}$ and $y = e^{-x^2}$.

58. Prove that $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx$ converges by comparing with $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx$ (Figure 11).

59. Show that $\displaystyle\int_1^\infty \frac{1-\sin x}{x^2}\,dx$ converges.

60. Let $a > 0$. Recall that $\displaystyle\lim_{x\to\infty} \frac{x^a}{\ln x} = \infty$ (by Exercise 66 in Section 4.5).

(a) Show that $x^a > 2\ln x$ for all x sufficiently large.

(b) Show that $e^{-x^a} < x^{-2}$ for all x sufficiently large.

(c) Show that $\displaystyle\int_1^\infty e^{-x^a}\,dx$ converges.

In Exercises 61–75, use the Comparison Test to determine whether or not the integral converges.

61. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^5+2}}\,dx$

62. $\displaystyle\int_1^\infty \frac{dx}{(x^3+2x+4)^{1/2}}$

63. $\displaystyle\int_3^\infty \frac{dx}{\sqrt{x}-1}$

64. $\displaystyle\int_0^5 \frac{dx}{x^{1/3}+x^3}$

65. $\displaystyle\int_1^\infty e^{-(x+x^{-1})}\,dx$

66. $\displaystyle\int_0^1 \frac{|\sin x|}{\sqrt{x}}\,dx$

67. $\displaystyle\int_0^1 \frac{e^x}{x^2}\,dx$

68. $\displaystyle\int_1^\infty \frac{1}{x^4+e^x}\,dx$

69. $\displaystyle\int_0^1 \frac{1}{x^4+\sqrt{x}}\,dx$

70. $\displaystyle\int_1^\infty \frac{\ln x}{\sinh x}\,dx$

71. $\displaystyle\int_5^\infty \frac{1}{x^2\ln x}\,dx$

72. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^{1/3}+x^3}}$

73. $\displaystyle\int_0^1 \frac{dx}{(8x^2+x^4)^{1/3}}$

74. $\displaystyle\int_1^\infty \frac{dx}{(x+x^2)^{1/3}}$

75. $\displaystyle\int_0^1 \frac{dx}{xe^x+x^2}$

Hint for Exercise 74: Show that for $x \ge 1$,

$$\frac{1}{(x+x^2)^{1/3}} \ge \frac{1}{2^{1/3}x^{2/3}}$$

Hint for Exercise 75: Show that for $0 \le x \le 1$,

$$\frac{1}{xe^x+x^2} \ge \frac{1}{(e+1)x}$$

76. Use the Comparison Test to determine for what values of p this integral converges: $\displaystyle\int_5^\infty \frac{1}{x^p\ln x}\,dx$.

77. Define $J = \displaystyle\int_0^\infty \frac{dx}{x^{1/2}(x+1)}$ as the sum of the two improper integrals

$$\int_0^1 \frac{dx}{x^{1/2}(x+1)} + \int_1^\infty \frac{dx}{x^{1/2}(x+1)}$$

Use the Comparison Test to show that J converges.

78. Determine whether $J = \displaystyle\int_0^\infty \frac{dx}{x^{3/2}(x+1)}$ (defined as in Exercise 77) converges.

79. An investment pays a dividend of \$250/year continuously forever. If the interest rate is 7%, what is the present value of the entire income stream generated by the investment?

80. An investment is expected to earn profits at a rate of $10{,}000e^{0.01t}$ dollars per year forever. Find the present value of the income stream if the interest rate is 4%.

81. Compute the present value of an investment that generates income at a rate of $5000te^{0.01t}$ dollars per year forever, assuming an interest rate of 6%.

82. Find the volume of the solid obtained by rotating the region below the graph of $y = e^{-x}$ about the x-axis for $0 \le x < \infty$.

83. When a capacitor of capacitance C is charged by a source of voltage V, the power expended at time t is

$$P(t) = \frac{V^2}{R}(e^{-t/RC} - e^{-2t/RC})$$

where R is the resistance in the circuit. The total energy stored in the capacitor is

$$W = \int_0^\infty P(t)\,dt$$

Show that $W = \frac{1}{2}CV^2$.