Chapter 6
Linear Programming: The Simplex Method

Section 1
A Geometric Introduction to the Simplex Method

6.1 Geometric Introduction to the Simplex Method

The geometric method of linear programming from the previous section is limited in that it is only useful for problems involving two decision variables and cannot be used for applications involving three or more decision variables. It is for this reason that a more sophisticated method must be developed.

George Dantzig
1914 - 2005

George B. Dantzig developed such a method in 1947 while being assigned to the U.S. military. Ideally suited to computer use, the method is used routinely on applied problems involving hundreds and even thousands of variables and problem constraints.

An Interview with George Dantzig, Inventor of the Simplex Method

http://www.e-optimization.com/directory/trailblazers/dantzig/interview_opt.cfm

IRV
How do you explain optimization to people who haven't heard of it?

GEORGE
I would illustrate the concept using simple examples such as the diet problem or the blending of crude oils to make high-octane gasoline.

IRV
What do you think has held optimization back from becoming more popular?

GEORGE
It is a technical idea that needs to be demonstrated over and over again. We need to show that firms that use it make more money than those who don't.

IRV
Can you recall when optimization started to become used as a word in the field?

GEORGE
From the very beginning of linear programming in 1947, terms like maximizing, minimizing, extremizing, optimizing a linear form and optimizing a linear program were used.
An Interview with George Dantzig (continued)

GEORGE
The whole idea of objective function, which of course optimization applies, was not known prior to linear programming. In other words, the idea of optimizing something was something that nobody could do, because nobody tried to optimize. So while you are very happy with it and say it's a very familiar term, optimization just meant doing it better than somebody else. And the whole concept of getting the optimum solution just didn't exist. So my introducing the whole idea of optimization in the early days was novel.

IRV
I understand that while programming the war effort in World War II was done on a vast scale, the term optimization as a word was never used. What was used instead?

GEORGE
A program can be thought of as a set of blocks, or activities, of different shapes that can be fitted together according to certain rules, or mass balance constraints. Usually these can be combined in many different ways, some more, some less desirable than other combinations. Before linear programming and the simplex method were invented, it was not possible to computationally determine the best combination such as finding the program that maximizes the number of sorties flown. Instead, all kinds of ground rules were invented deemed by those in charge to be desirable characteristics for a program to have.

IRV
Name some of your most important early contributions.

GEORGE
The first was the recognition that most practical planning problems could be reformulated mathematically as finding a solution to a system of linear inequalities. My second contribution was recognizing that the plethora of ground rules could be eliminated and replaced by a general objective function to be optimized. My third contribution was the invention of the simplex method of solution.

IRV
And these were great ideas that worked and still do.

GEORGE
Yes, I was very lucky.

IRV
What would you say is the most invalid criticism of optimization?

GEORGE
Saying: "It's a waste of time to optimize because one does not really know what are the exact values of the input data for the program."

IRV
Ok, let's turn this around. What would you say is the greatest potential of optimization?

GEORGE
It has the potential to change the world.

An Interview with George Dantzig (continued)

Standard Maximization Problem in Standard Form

A linear programming problem is said to be a standard maximization problem in standard form if its mathematical model is of the following form:

Maximize

$$P = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to problem constraints of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b, \quad b \geq 0$$

with nonnegative constraints $x_1, x_2, \ldots, x_n \geq 0$
Example

We will use a modified form of a previous example. Consider the linear programming problem of maximizing $z$ under the given constraints. This is a standard maximization problem.

$$z = 5x + 10y$$
$$8x + 8y \leq 160$$
$$4x + 12y \leq 180$$
$$x \geq 0; \ y \geq 0$$

Example (continued)

There are four lines bordering the feasible region:

- $8x + 8y = 0$
- $4x + 12y = 0$
- $x = 0$
- $y = 0$

Any two of them intersect in a point. There are a total of six such points.

The coordinates of the point can be found by solving the system of two equations in two unknowns created by the equations of the two lines.

Some of the points are corner points of the feasible region, and some are outside.

Example (continued)

grid spacing = 5 units

$$z = 5x + 10y$$
$$8x + 8y \leq 160$$
$$4x + 12y \leq 180$$
$$x \geq 0; \ y \geq 0$$

| $(0,0)$ | feasible |
| $(0,15)$ | feasible |
| $(0,20)$ | not feasible |
| $(7.5,12.5)$ | feasible (optimal) |
| $(20,0)$ | feasible |
| $(45,0)$ | not feasible |

Slack Variables

To use the simplex method, the constraint inequalities must be converted to equalities. Consider the two constraint inequalities

$$8x + 8y \leq 160$$
$$4x + 12y \leq 180$$

To make this into a system of equations, we introduce slack variables

$$8x + 8y + s_1 = 160$$
$$4x + 12y + s_2 = 180$$

They are called slack variables because they take up the slack between the left and right hand sides of the inequalities.

Note that the slack variables must be nonnegative.
Slack Variables (continued)

- We now have two equations in four unknowns \(x, y, s_1, s_2\).
- The system has an infinite number of solutions since there are more unknowns than equations. We can make the system have a unique solution by assigning two of the variables a value of zero and then solving for the remaining two variables. This is called a **basic solution**.
- We select any two of the four variables as **basic variables**. The remaining two variables automatically become **non-basic variables**. The non-basic variables are always assigned a value of zero. Then we solve the equations for the two basic variables.

Example (continued)

The equations are

\[
\begin{align*}
8x + 8y + s_1 &= 160 \\
4x + 12y + s_2 &= 180
\end{align*}
\]

Select \(x = 0\) and \(y = 0\) as the non-basic variables. Then the first equation reads \(8(0) + 8(0) + s_1 = 160\), so \(s_1 = 160\). The second equation becomes \(4(0) + 12(0) + s_2 = 180\), so \(s_2 = 180\).

This corresponds to the point \((x, y) = (0, 0)\). According to the graph we saw earlier, this is a feasible point.

<table>
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<tr>
<th>(x)</th>
<th>(y)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>point</th>
<th>feasible?</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>0</td>
<td>160</td>
<td>180</td>
<td>(0,0)</td>
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</tr>
<tr>
<td>0</td>
<td>20</td>
<td>0</td>
<td>-60</td>
<td>(0,20)</td>
<td>no</td>
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<tr>
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<td>15</td>
<td>40</td>
<td>0</td>
<td>(0,15)</td>
<td>yes</td>
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<tr>
<td>20</td>
<td>0</td>
<td>0</td>
<td>100</td>
<td>(20,0)</td>
<td>yes</td>
</tr>
<tr>
<td>45</td>
<td>0</td>
<td>0</td>
<td>-200</td>
<td>(45,0)</td>
<td>no</td>
</tr>
<tr>
<td>7.5</td>
<td>12.5</td>
<td>0</td>
<td>0</td>
<td>(7.5,12.5)</td>
<td>yes</td>
</tr>
</tbody>
</table>

The other 5 choices of non-basic variables are done similarly. The table on the next slide summarizes the results.

Discovery!

- Each basic solution corresponds to an intersection point of the boundary lines of the feasible region.
- A feasible basic solution corresponds to a corner point (vertex) of the feasible region. This includes the optimum solution of the linear programming problem.
- A basic solution that is not feasible includes at least one negative value; a basic feasible solution does not include any negative values.
- We can determine the feasibility of a basic solution simply by examining the signs of all the variables in the solution.
Generalization

- In a linear programming problem with slack variables there will always be more variables than equations.
- Basic variables are selected arbitrarily (as many basic variables as there are equations). The remaining variables are called non-basic variables.
- We obtain a basic solution by assigning the non-basic variables a value of zero and solving for the basic variables. If a basic solution has no negative values, it is basic feasible solution.
- Theorem: If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one (or more) of the basic feasible solutions.

Conclusion

This is simply the first step in developing a procedure for solving a more complicated linear programming problem. But it is an important step in that we have been able to identify all the corner points (vertices) of the feasible set without having three or more variables.