APPLICATIONS OF DIFFERENTIATION

4.9 Antiderivatives

Objective: Find a function whose derivative is a known function.

I. A function F is called an ***antiderivative*** of f on an interval I if $F'(x) = f(x)$ for all x in I.

A. Suppose $f(x) = 3x^2$

1. $F(x) = x^3$ is an antiderivative of f(x) because $F'(x) = 3x^2 = f(x)$.
2. $F(x) = x^3 + p$ is an antiderivative of f(x) because $F'(x) = 3x^2 = f(x)$.
3. $F(x) = x^3 - 4e^5$ is an antiderivative of f(x) because $F'(x) = 3x^2 = f(x)$.
4. $F(x) = x^3 + C$ is the most general antiderivative of f(x), where C is an arbitrary constant

B. Suppose $f(x) = x^n$

1. $F(x) = \frac{1}{5}x^5 + C$ is the most general antiderivative of $F(x) = x^4$ because

$$F'(x) = 5\left(\frac{1}{5}x^{5-1}\right) = x^4 = f(x).$$

2. $F(x) = \frac{1}{n+1}x^{n+1} + C$ is the most general antiderivative of $f(x) = x^n$

because $F'(x) = (n+1)\left(\frac{1}{n+1}x^{(n+1)-1}\right) = x^n = f(x)$.

C. Antidifferentiation formulas

Function	**A particular antiderivative**	**The general antiderivative**
cf(x)	cF(x) + 5	cF(x) + C
f(x) + g(x)	F(x) + G(x) - 7e	F(x) + G(x) + C
$x^n,\ n \neq 1$	$\frac{x^{n+1}}{n+1}$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	ln | x |	ln | x | + C

e^x	$e^x - \ln 4$	$e^x + C$
$\cos x$	$\sin x$	$\sin x + C$
$\sin x$	$-\cos x$	$-\cos x + C$
$\sec^2 x$	$\tan x$	$\tan x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$\sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} x$	$\tan^{-1} x + C$

D. Find the [general] antiderivative of

1. $f(x) = 2 + \sqrt[5]{x^3} = 2 + x^{3/5}$

   $F(x) = 2x + \frac{5}{8}x^{8/5} + C$

2. $f(x) = 5\sec^2 x + 7x^2 + 4x^{3/5} - \frac{e}{x} + 1$

   $F(x) = 5\tan x + \frac{1}{3}7x^3 + \frac{5}{8}4x^{8/5} - e\ln|x| + x + C$

   $F(x) = 5\tan x + \frac{7}{3}x^3 + \frac{5}{2}x^{8/5} - e\ln|x| + x + C$

3. Remember to check your work by taking the derivative of your answer; this should get you back to the original problem!

II. An equation that involves the derivative(s) of a function is called a ***differential equation***.

A. Find f(x) if $f'(x) = e^x + \frac{20}{1+x^2}$ and $f(0) = -2$

1. $f(x) = e^x + 20\tan^{-1} x + C$ is the ***general solution***.
2. $f(0) = e^0 + 20\tan^{-1} 0 + C = -2 \rightarrow \quad C = -2 - 1 - 0 = -3$
3. $f(x) = e^x + 20\tan^{-1} x - 3$ is the ***particular solution.***

B. Find f(x) if $f''(x) = 12x^2 + 6x - 4$, if $f(0) = 4$ and $f(1) = 1$

1. $f'(x) = 12 \bullet \frac{x^3}{3} + 6 \bullet \frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$

2. $f(x) = 4\bullet\frac{x^4}{4} + 3\bullet\frac{x^3}{3} - 4\bullet\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$

   a. $f(0) = 0 + 0 - 0 + 0 + D = 4 \rightarrow D = 4$
   b. $f(1) = 1 + 1 - 2 + C + 4 = 1 \rightarrow C = -3$
   c. $f(x) = x^4 + x^3 - 2x^2 + 4x - 3$

III. Sketch the graph of F(x) if $f(x) = \sqrt{1+x^3} - x$ and $F(-1) = 0$

A. We do not know how to find an antiderivative of f(x).
B. We could draw the graph of f and use it to approximate the graph of F(x) [see Example 4 in Sec 2.10]
C. See page 336 for the direction field for f(x)

   1. Since f(0) = 1, the graph of F(x) has slope 1 when x = 0; draw a short tangent segment with slope 1, centered at x = 0.
   2. Since $f(1) \approx .4$, the graph of F(x) has slope $.4$ when x = 1; draw a short tangent segment with slope $.4$, centered at x = 1.
   3. Repeat this process for several other values of x.
   4. Each segment in the direction field indicates the direction in which the curve y = F(x) proceeds at that point.

D. The initial condition $F(-1) = 0$ means that we should start at the point $(-1, 0)$ and draw the graph of F(x) so that it follows the directions of the tangent segments.

IV. Rectilinear motion

A. If s = f(t) is the position function of a particle, then the velocity function is v(t) = $s'(t)$; the position function is an antiderivative of the velocity function!
B. If v = v(t) is the velocity function, then the acceleration function is a(t) = $v'(t)$; the velocity function is an antiderivative of the acceleration function.

V. Example 7 on p. 337

A ball is thrown upward with a speed of 48 ft/s from a height of 432 ft.

A. Find its height above the ground t seconds later.

   1. Motion is vertical; choose upward to be the positive .direction
   2. The only acceleration is the force of gravity, which is pulling downward

      1. $a(t) = \frac{dv}{dt} = -32\ ft/s^2$
      2. $v(t) = -32t + C$ and $v_0 = 48 \rightarrow C = 48$
      3. $v(t) = -32t + 48$

B. When does it reach its maximum height?

1. Maximum height $\rightarrow$ $v(t) = 0$
2. $-32t + 48 = 0 \rightarrow t = 1.5$ s

C. When does it hit the ground?

1. $v(t) = \dfrac{ds}{dt} = -32t + 48$
2. $s(t) = -16t^2 + 48t + D$ and $s(0) = 432 \rightarrow D = 432$
3. $s(t) = -16t^2 + 48t + 432$
4. When the ball hits the ground s(t) = 0, so $-16t^2 + 48t + 432 = 0$, so $t = \dfrac{3 \pm 3\sqrt{13}}{2}$
5. The ball hits the ground after approximately 6.9 seconds.