INTEGRALS

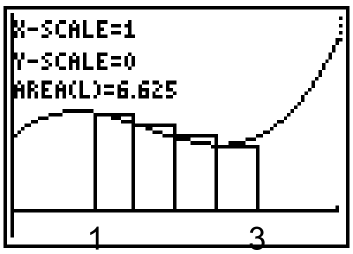
5.1 Areas and Distances

Objectives: Find area under a curve and total distance traveled by a car

I. Estimate the area under the graph of $f(x) = .5x^3 - 2.5x^2 + 3x + 3$ (and above the x-axis) from x = 1 to x = 3 using 4 rectangles.

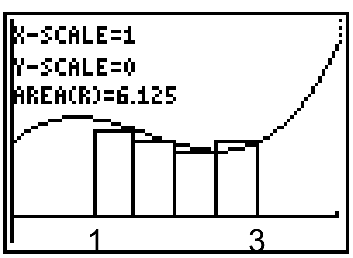
A. The width of each subinterval = $\Delta x = \dfrac{b-a}{n} = \dfrac{3-1}{4} = \dfrac{1}{2}$

B. Using left endpoints: height of each rectangle = function evaluated at left endpoint.



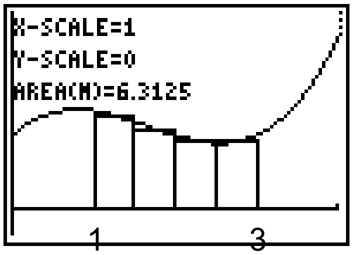
$$A \approx L_4 = \Delta x \bullet f(1) + \Delta x \bullet f(1.5) + \Delta x \bullet f(2) + \Delta x \bullet f(2.5)$$
$$= \frac{1}{2}\left(4 + \frac{57}{16} + 3 + \frac{43}{16}\right) = 6.625$$

C. Using right endpoints: height of each rectangle = function evaluated at right endpoint.



$$A \approx R_4 = \Delta x \bullet f(1.5) + \Delta x \bullet f(2) + \Delta x \bullet f(2.5) + \Delta x \bullet f(3)$$
$$= \frac{1}{2}\left(\frac{57}{16} + 3 + \frac{43}{16} + 3\right) = 6.125$$

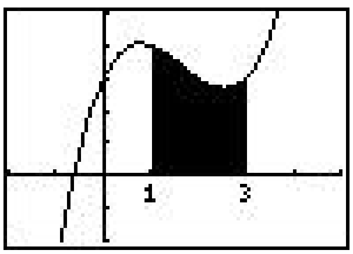
D. Using midpoints: height of each rectangle = function evaluated at midpoint.

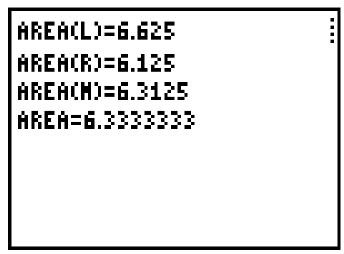


$$A \approx M_4 = \Delta x \bullet f(1.25) + \Delta x \bullet f(1.75) + \Delta x \bullet f(2.25) + \Delta x \bullet f(2.75)$$

$$= \frac{1}{2}\left(\frac{489}{128} + \frac{419}{128} + \frac{357}{128} + \frac{351}{128}\right) = 6.3125$$

E. Exact area





F. The estimate can be improved by using a larger value for n.

II. Find the area under the graph of $y = x^2$ from x = 0 to x = 1

A. $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$

B. $R_n = \dfrac{1}{n}\left(\dfrac{1}{n}\right)^2 + \dfrac{1}{n}\left(\dfrac{2}{n}\right)^2 + \dfrac{1}{n}\left(\dfrac{3}{n}\right)^2 + \dfrac{1}{n}\left(\dfrac{4}{n}\right)^2 + \ldots + \dfrac{1}{n}\left(\dfrac{n}{n}\right)^2$

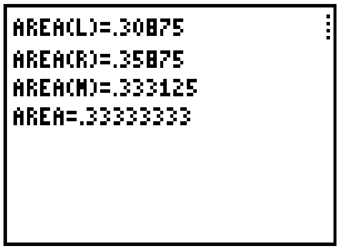
$R_n = \dfrac{1}{n^3}\left(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2\right)$

C. Use *mathematical induction* to prove that

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

D. $A = \lim\limits_{n \to \infty} \dfrac{1}{n^3}\left(\dfrac{n(n+1)(2n+1)}{6}\right) = \lim\limits_{n \to \infty}\left(\dfrac{(n+1)(2n+1)}{6n^2}\right)$

$$= \lim_{n \to \infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = \lim_{n \to \infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}(1)(2) = \frac{1}{3}$$

E. 

III. Finding an infinite sum

Find $\sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \ldots + (n-2) + (n-1) + n$

A. Case 1: Suppose n is *even*

1. Then $\sum_{k=1}^{n} k = (1+n) + [2+(n-1)] + [3+(n-2)] + [4+(n-3)] + \ldots$

   a. There are $\frac{n}{2}$ terms

   b. Each term = n + 1

   c. $\sum_{k=1}^{n} k = \frac{n}{2}(n+1) = \frac{n(n+1)}{2}$, if n is even

B. Case 2: Suppose n is *odd*

1. Then $\sum_{k=1}^{n} k = (1+n) + [2+(n-1)] + [3+(n-2)] + \ldots + MT$

2. There are $\frac{n-1}{2}$ terms of (n + 1); middle term = $\frac{n+1}{2}$

3. $\sum_{k=1}^{n} k = \left(\frac{n-1}{2}\right)(n+1) + \left(\frac{n+1}{2}\right) = \frac{n(n+1)}{2}$, if n is odd

C. It *appears* that the sum of the first n integers is always $\frac{n(n+1)}{2}$, but we have not proved it!

IV. Proof by mathematical induction - see handout on my web site

Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Let S_n be the statement you are trying to prove, i.e. $S_n = \frac{n(n+1)}{2}$

A. *Prove* that S_n is true for the first 2 or 3 integers

1. If n = 1, then $\sum_{k=1}^{1} k = \frac{1(1+1)}{2} = 1$ and 1 = 1, so S_1 is true.
2. If n = 2, then $\sum_{k=1}^{2} k = \frac{2(2+1)}{2} = 3$ and 1 + 2 = 3, so S_2 is true.
3. If n = 3, then $\sum_{k=1}^{3} k = \frac{3(3+1)}{2} = 6$ and 1 + 2 + 3 = 6, so S_3 is true.

B. *Assume* that S_k is true for some integer $j \geq 1$.

That is, $\sum_{k=1}^{j} k = \frac{j(j+1)}{2}$ is assumed to be true.

C. *Prove* that S_{k+1} is true whenever S_k is true

$$\sum_{k=1}^{j+1} k = \frac{j(j+1)}{2} + (j+1) = \frac{j(j+1)}{2} + \frac{2(j+1)}{2} = \frac{(j+1)(j+2)}{2}$$

$$= \frac{(j+1)[(j+1)+1]}{2} = \frac{n(n+1)}{2}, \text{ where } n = j + 1$$

D. $\therefore S_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ is true for all positive integers n

V. Find the *total distance* traveled by the car in Example 4 on p. 357

A. Note that conversion of units is necessary in this problem

$$17\frac{mi}{hr}\bullet\frac{1hr}{60s}\bullet\frac{5280ft}{1mi} = 25ft/s$$

B. Each time period is 5 s long; we will use the velocity at the beginning of each time period

Distance traveled = rt = $25 \bullet 5 + 31 \bullet 5 + 35 \bullet 5 + 43 \bullet 5 + 47 \bullet 5 + 46 \bullet 5$

= 1135 ft

C. Using the velocity at the end of each time period gives 1215 ft