INTEGRALS

5.2 The Definite Integral

Objective: Evaluate definite integrals using Riemann sums

I. If f is continuous on [a, b], then the definite integral of f from a to b is

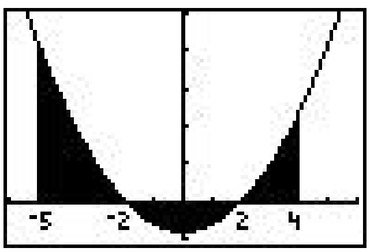
$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a *Riemann sum*

II. Worksheet on summations

III. Use the Midpoint Rule with $\Delta x = \frac{1}{2}$ to approximate $\int_{-5}^{4} (x^2 - 4)dx$

A.



1. $\int_{-5}^{-2} (x^2 - 4)dx \approx M_6 = 26.9375$
2. $\int_{-2}^{2} (x^2 - 4)dx \approx M_8 = -10.75$ NOTE: If area is below the x-axis, the integral will be negative!
3. $\int_{2}^{4} (x^2 - 4)dx \approx M_4 = 10.625$
4. $\int_{-5}^{4} (x^2 - 4)dx \approx M_{18} = 26.8125$

IV. Use Riemann sums to estimate the value of $\int_0^4 (x^2 - 4)dx$

$$\int_0^4 (x^2 - 4)dx \approx \frac{1}{2}\int_{-2}^{2} (x^2 - 4)dx + \int_2^4 (x^2 - 4)dx = \frac{1}{2}(-10.75) + 10.625 = 5.25$$

V. Use summations to find the exact value of $\int_0^4 (x^2 - 4)dx$

A. $\Delta x = \frac{b - a}{n} = \frac{4}{n}$

B. $x_0 = 0$, $x_1 = \frac{4}{n}$, $x_2 = \frac{8}{n}$, $x_3 = \frac{12}{n}$, and, in general, $x_i = \frac{4i}{n}$.

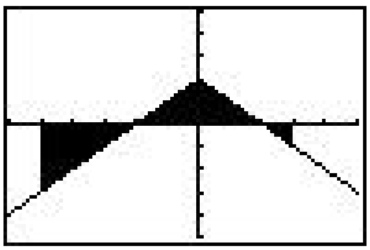
C. $\int_0^4 (x^2 - 4)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\frac{4}{n}$

$= \lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \left[\left(\frac{4i}{n}\right)^2 - 4\right] = \lim_{n \to \infty} \frac{4}{n} \sum_{i=1}^{n} \left[\frac{16}{n^2} i^2 - 4\right]$

$= \lim_{n \to \infty} \left[\frac{64}{n^3} \sum_{i=1}^{n} i^2 - \frac{4}{n} \sum_{i=1}^{n} 4\right] = \lim_{n \to \infty} \left\{\frac{64}{n^3} \left[\frac{n(n+1)(2n+1)}{6}\right] - \frac{4}{n}(4n)\right\}$

$= \lim_{n \to \infty} \left\{\frac{64}{6} \left[\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] - 16\right\} = \frac{32}{3}(1 \cdot 2) - 16 = \frac{16}{3}$

V. Evaluate $\int_{-5}^{3} \left(2 - |x|\right)dx$ by interpreting it in terms of areas.

A. 

First find the zeros of the function on the required interval; they are -2 and 2.

B. $\int_{-5}^{-2} \left(2 - |x|\right)dx = -\left(\frac{1}{2} \cdot 3 \cdot 3\right) = -\frac{9}{2}$

C. $\int_{-2}^{2} \left(2 - |x|\right)dx = \left(\frac{1}{2} \cdot 4 \cdot 2\right) = 4$

D. $\int_{2}^{3} \left(2 - |x|\right)dx = -\left(\frac{1}{2} \cdot 1 \cdot 1\right) = -\frac{1}{2}$

E. $\int_{-5}^{3} \left(2 - |x|\right)dx = \int_{-5}^{-2} \left(2 - |x|\right)dx + \int_{-2}^{2} \left(2 - |x|\right)dx + \int_{2}^{3} \left(2 - |x|\right)dx$

$= -\frac{9}{2} + 4 - \frac{1}{2} = -1$