INTEGRALS

5.3 Evaluating Definite Integrals

Objective: Evaluate definite integrals using the evaluation theorem

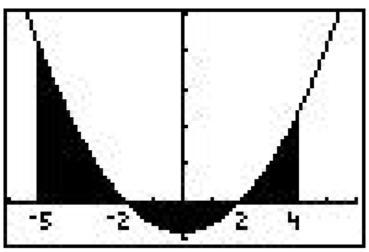
I. If f is continuous on [a, b], then the definite integral of f from x = a to x = b is $\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = F(b) - F(a)$, where F is ***any*** antiderivative of f.

Since F can be any antiderivative, we may choose to let C = 0!

II. Evaluation theorem: $\int_a^b f(x)dx = F(b) - F(a)$

III. Find $\int_{-5}^{4} (x^2 - 4)dx$ and interpret the result in terms of areas.

A. $\int_{-5}^{4} (x^2 - 4)dx = \left[\frac{x^3}{3} - 4x\right]_{-5}^{4} = \left[\frac{4^3}{3} - 4(4)\right] - \left[\frac{(-5)^3}{3} - 4(-5)\right] = 27$

B. 

1. $\int_{-5}^{-2} (x^2 - 4)dx \approx M_6 = 26.9375$

$\int_{-5}^{-2} (x^2 - 4)dx = \left[\frac{(-2)^3}{3} - 4(-2)\right] - \left[\frac{(-5)^3}{3} - 4(-5)\right] = 27$

2. $\int_{-2}^{2} (x^2 - 4)dx \approx M_8 = -10.75$

$\int_{-2}^{2} (x^2 - 4)dx = \left[\frac{(2)^3}{3} - 4(2)\right] - \left[\frac{(-2)^3}{3} - 4(-2)\right] = -\frac{32}{3}$

3. $\int_{2}^{4} (x^2 - 4)dx \approx M_4 = 10.625$

$\int_{2}^{4} (x^2 - 4)dx = \left[\frac{(4)^3}{3} - 4(4)\right] - \left[\frac{(2)^3}{3} - 4(2)\right] = \frac{32}{3}$

4. $\int_{-5}^{4}(x^2 - 4)dx \approx M_{18} = 26.8125$

$$\int_{-5}^{4}(x^2 - 4)dx = 27 - \frac{32}{3} + \frac{32}{3} = 27$$

C. The integral equals the sum of the areas above the x-axis minus the areas below the x-axis

IV. Find the exact value of $\int_{0}^{4}(x^2 - 4)dx$

A. Applying equation 3 from Section 5.2 [p. 358]

$$\int_{0}^{4}(x^2 - 4)dx = \frac{16}{3}$$

B. Using the evaluation theorem from Section 5.3 [p.369]

$$\int_{0}^{4}(x^2 - 4)dx = \left[\frac{x^3}{3} - 4x\right]_0^4 = \left[\frac{(4)^3}{3} - 4(4)\right] - \left[\frac{(0)^3}{3} - 4(0)\right] = \frac{16}{3}$$

V. The ***indefinite*** integral $\int f(x)dx$ is a ***function***.

A. The ***indefinite*** integral $\int f(x)dx$ is the set of all ***antiderivatives*** of the function F(x), where $f'(x) = F(x)$.

B. Example: $\int \frac{1}{x}dx = \ln|x| + C$

1. C is called the ***constant of integration***.
2. C may be any real number, but can only be evaluated if an ***initial condition*** is given

VI. The ***definite*** integral $\int_a^b f(x)dx$ is a ***number***.

Example: $\int_1^e \frac{1}{x}dx = \left[\ln|x|\right]_1^e = \ln(e) - \ln(1) = 1$

VII. ***KNOW*** the table of indefinite integrals on page 372

A. These formulas are only valid on closed intervals where the function is ***<u>continuous</u>***

B. $\int f(x)dx = F(x) \Leftrightarrow F'(x) = f(x)$

VIII. Total change theorem

A. If y = F(x), then $F'(x)$ represents the ***rate of change*** of F(x) with respect to x, and F(b) – F(a) is the ***total change*** in y when x changes from a to b

B. $\int_a^b F'(x)dx = F(b) - F(a)$

IX. The (ever-popular) moving particle problem: A particle moves along a line with velocity of $v(t) = (t^2 - t - 6)\,m/s$ at time t.

A. Find the displacement of the particle from t = 1 s to t = 4 s

1. Displacement = change in location relative to starting point

$$s(4) - s(1) = \int_1^4 v(t)dt = \int_1^4 (t^2 - t - 6)dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2} = -4.5m$$

2. This means the particle is now located 4.5 m ***left*** of its starting point

B. Find the total distance traveled during this time period

1. Recall that speed = $|v(t)| = |(t-3)(t+2)|$
   $v(t) < 0$ on (1, 3) and $v(t) > 0$ on (3, 4)

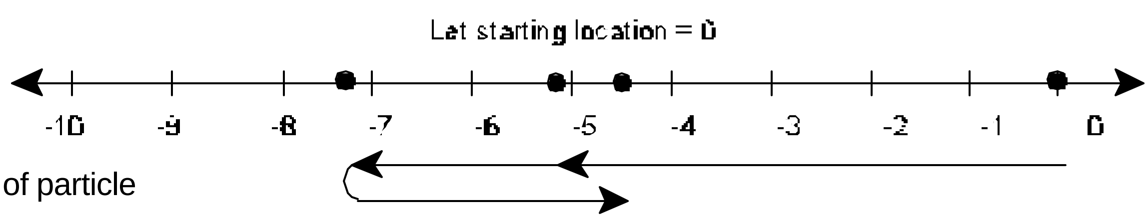
2. Total distance traveled = $\int_1^4 |v(t)|dt = -\int_1^3 v(t)dt + \int_3^4 v(t)dt$

$$= -\int_1^3 (t^2 - t - 6)dt + \int_3^4 (t^2 - t - 6)dt$$

$$= -\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 = \frac{61}{6} \approx 10.17m$$

3. This means the particle has moved a total of approximately 10.17 m during the time the period from time t = 1 s to t = 4 s.

C. Visualization of the motion of the particle



path of particle

time t in sec	3	2	4	1
displacement in m	-22/3	-31/6	-4.5	0
velocity in m/s	0	-4	6	-6

X. General motion of an object

A. If an object moves along a straight line with position function s(t) at time t,

1. Then its velocity is $v(t) = s'(t)$ at time t, (speed = | v(t) |)
2. And $\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$ is the ***total change of position***, or ***displacement***, of the particle during the period from time t_1 to time t_2.

B. If the velocity function of the object is v(t) at time t,

1. Then its ***acceleration*** (change in velocity) is $a(t) = v'(t)$ at time t,
2. And $\int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1)$ is the ***total change*** in velocity from time t_1 to time t_2.

XI. Applications of the integral to find total change (see p. 374-5)

A. If V(t) = volume of water in a reservoir at time t,

1. Then $V'(t)$ is the ***rate*** at which water flows into (or out of) the reservoir at time t,
2. And $\int_{t_1}^{t_2} V'(t)dt = V(t_2) - V(t_1)$ is the ***total change*** in the amount of water in the reservoir between time t_1 and time t_2

B. If [C](t) is the concentration of the product C of a chemical reaction at time t

1. Then the ***rate of reaction*** is $\frac{d}{dt}[C]$ at time t,
2. And $\int_{t_1}^{t_2} \frac{d[C]}{dt} = [C](t_2) - [C](t_1)$ is the ***total change*** in the concentration of C from time t_1 to time t_2.

C. If the mass of a rod measured from the left end to point x is m(x),

1. Then the ***linear density*** (change in mass per unit length) is $\rho(x) = m'(x)$ at point x,
2. And $\int_a^b \rho(x)dx = m(b) - m(a)$ is the ***total mass*** of the segment of the rod that lies between x = a and x = b.

D. If n(t) represents a population at time t,

1. Then the ***rate of growth*** (or decay) of the population is $\frac{dn}{dt}$ at time t,

2. And $\int_{t_1}^{t_2} \frac{dn}{dt} = n(t_2) - n(t_1)$ is the ***total change*** in the population during the period period from time t_1 to time t_2.

E. If C(x) is the cost of producing x units of a commodity,

1. Then the ***marginal cost*** of producing the (x+1)st unit is $C'(x)$,

2. And $\int_{x_1}^{x_2} \frac{dC}{dt} = C(x_2) - C(x_1)$ is the total increase in cost when production is increased from x_1 units to x_2 units.

F. Power is the ***rate of change*** of energy: P(t) = $E'(t)$.

XII. The unit of measurement for $\int_a^b f(x)dx$ is the product of the unit for f(x) and the unit for x

Example: A particle moves along a line with velocity of v(t) = $(t^2 - t - 6)m/s$ at time t.

Displacement = s(4) – s(1) = $\int_1^4 v(t)dt = \int_1^4 (t^2 - t - 6)dt = -4.5m$,

because v(t) is in m/s and dt is in s, therefore the product is in $\frac{m}{s}(s) = m$.