INTEGRALS

5.4 The Fundamental Theorem of Calculus

Objective: Investigate the link between differentiation and integration as given by the Fundamental Theorem of Calculus

I. Some properties of integrals

A. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

B. If a = b, then $\Delta x = 0$, so $\int_a^a f(x)dx = 0$

II. Comparison properties of integrals

A. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$

This can be interpreted as the area under the graph of a function where the area is entirely above the x-axis.

B. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

This means that a “bigger” function has a “bigger” integral.

C. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$

The area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M

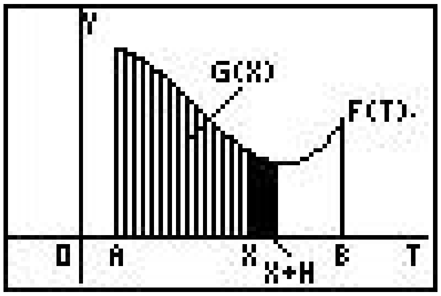
III. Motivation for the fundamental theorem: let f be a continuous function on [a, b] and define a function g by $g(x) = \int_a^x f(t)dt$ where $a \leq x \leq b$.

A. g depends only on x

1. If x is a fixed number, then $\int_a^x f(t)dt$ is a definite number.
2. If x is a variable, then $\int_a^x f(t)dt$ also varies and defines a function of x, denoted by g(x)
3. Example: Suppose $f(t) = t^2$ and a = 1
    a. $g(x) = \int_1^x t^2 dt = \dfrac{x^3 - 1}{3}$
    b. $g'(x) = x^2$, that is $g' = f$
4. If g is defined as the integral of f, then g turns out to be an antiderivative of f

(in this case).

B. In general, consider any continuous function f with $f(x) \geq 0$.



1. $g(x) = \int_a^x f(t)dt$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b.
2. For $h > 0$, $g(x + h) - g(x)$ is the area under the graph of f from x to $(x + h)$.

   a. If h is small, this area is approximately equal to the area of the rectangle with height $f(x)$ and width h, that is $g(x + h) - g(x) \approx h[f(x)]$.

   b. Then $\dfrac{g(x+h) - g(x)}{h} \approx f(x)$

3. It appears that $g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h} = f(x)$!

IV. The fundamental theorem of calculus: Suppose f is continuous on [a, b]

Part 1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$

If f is integrated and the result is then differentiated, we arrive back at the original function f.

Part 2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

If we differentiate a function F, and then integrate the result, we arrive back at the original function F in the form of $F(b) - F(a)$.

V. Proof of FTCII: Divide the interval [a, b] into n subintervals with endpoints x_0 (= a), x_1, x_2, x_3, . . . , x_n (= b) and with length $\Delta x = \dfrac{b - a}{n}$.

A. $F(b) - F(a) = F(x_n) - F(x_0)$

$$= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \dots + [F(x_2) - F(x_1)] + [F(x_1) - F(x_0)]$$

$$= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$$

B. Applying the Mean Value Theorem (see p. 283) to F on each subinterval $[x_i, x_{i-1}]$

1. There exists a number x_i^* between x_{i-1} and x_i such that

   $F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) = f(x_i^*)\Delta x$

2. Therefore, $F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\Delta x$

C. $\lim_{n \to \infty}[F(b) - F(a)] = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \to F(b) - F(a) = \int_a^b f(x)dx$

VI. Proof of FTCI: Let $g(x) = \int_a^x f(t)dt$

A. If x and (x + h) are in the open interval (a, b), then

$g(x + h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$

$= \left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt\right) - \int_a^x f(t)dt = \int_a^{x+h} f(t)dt$

B. Then, for $h \neq 0$, $\dfrac{g(x+h) - g(x)}{h} = \dfrac{1}{h}\int_a^{x+h} f(t)dt$

C. Assume that h > 0.

1. Since f is continuous on [x, x + h], there are numbers u and v in [x, x + h] such that f(u) = m and f(v) = M, where m and M are the absolute minimum and maximum values of f on [x, x + h].
2. Applying property 3 with b – a = (x + h) – x = h

   $mh \le \int_a^{x+h} f(t)dt \le Mh$, that is, $f(u)h \le \int_a^{x+h} f(t)dt \le f(v)h$

   $h > 0 \to f(u) \le \dfrac{1}{h}\int_a^{x+h} f(t)dt \le f(v)$

3. From above: $f(u) \le \dfrac{g(x+h) - g(x)}{h} \le f(v)$
4. There is a similar argument for h < 0.

D. Let $h \to 0$

1. Then $u \to x$ and $v \to x$, since u and v lie between x and x + h.
2. Thus, $\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x)$ and $\lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x)$ because f is continuous at x.
3. Applying the squeeze theorem (see p. 117):

   $g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h} = f(x)$

VII. Find $g'(x)$ if $g(x) = \int_0^x (1 + t^2)dt$

A. Using FTCI: $g(x) = \int_0^x (1 + t^2)dt \rightarrow g'(x) = f(x) = 1 + x^2$, or

B. Using FTXII: $g(x) = \int_0^x (1 + t^2)dt = \left[t + \frac{t^3}{3} \right]_0^x = x + \frac{x^3}{3} \rightarrow g'(x) = 1 + x^2$