INTEGRALS

5.5 The Substitution Rule

Objective: Evaluate integrals using the substitution rule to change variables

I. The substitution rule

If u = g(x) is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f[g(x)]g'(x)dx = f(u)du$$

II. Using substitution to find indefinite integrals: General steps

A. Choose the quantity which “u” will represent. You must be able to differentiate “u”: common choices include

1. Quantity which is raised to a power
2. Quantity which is the argument of a function
3. Higher power if multiple powers are involved
4. Exponent of “e”
5. $\ln x$

B. Find the differential du
C. Rewrite given integral in terms of u and du
D. Substitute, using a substitution factor if necessary
E. Integrate
F. Back substitute

III. Using substitution to find indefinite integrals: Examples

A. Evaluate $\int 12x^2\sqrt{4x^3+7}\,dx$ let $u = 4x^3 + 7$, then $du = 12x^2dx$

$$\int 12x^2\sqrt{4x^3+7}\,dx = \int (4x^3+7)^{1/2}(12x^2dx)$$

$$= \int u^{1/2}du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(4x^3+7)^{3/2} + C$$

B. Evaluate $\int \sqrt[3]{(5x-7)^4}\,dx = \int (5x-7)^{4/3}dx$ let $u = 5x + 7$, then $du = 5dx$

$$\frac{1}{5}\int (5x-7)^{4/3}(5dx) = \frac{1}{5}\int (u)^{4/3}du = \frac{1}{5}\left(\frac{3}{7}u^{7/3}\right) + C = \frac{3}{35}(5x-7)^{7/3} + C$$

C. Evaluate $\int 8\cos(4x)dx$ let $u = 4x$, then $du = 4dx$

$$\int 8\cos(4x)dx = 2\int \cos(4x)(4dx) = 2\int \cos(u)du = 2\sin u + C = 2\sin(4x) + C$$

D. Evaluate $\int 2x^2(2x^3+1)^7dx$ let $u = 2x^3+1$, then $du = 6x^2dx$

$$\int 2x^2(2x^3+1)^7dx = \frac{1}{3}(2x^3+1)^7(6x^2) = \frac{1}{3}\int u^7du = \frac{1}{3}\left(\frac{u^8}{8}\right) + C = \frac{(2x^3+1)^8}{24} + C$$

E. Evaluate $\int \frac{7(x^2-1)}{(x^3-3x+1)^6}dx$

let $u = x^3-3x+1$, then $du = (3x^2-3)dx$ or $du = 3(x^2-1)dx$

$$\int \frac{7(x^2-1)}{(x^3-3x+1)^6}dx = \frac{7}{3}\int \frac{1}{(x^3-3x+1)^6}[3(x^2-1)dx]$$

$$= \frac{7}{3}\int u^{-6}du = \frac{7}{3}\int\left(\frac{u^{-5}}{-5}\right) + C = \frac{-7}{15(x^3-3x+1)^5} + C$$

F. Evaluate $\int [\cos^3(5t)]\sin(5t)dt$ let $u = \cos(5t)$, then $du = -5\sin(5t)\,dt$

$$\int [\cos^3(5t)]\sin(5t)dt = -\frac{1}{5}\int [\cos(5t)]^3[-5\sin(5t)dt] = -\frac{1}{5}\int u^3du$$

$$= -\frac{1}{5}\left(\frac{u^4}{4}\right) + C = -\frac{1}{20}\cos^4(5t) + C$$

G. Evaluate $\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx$ let $u = \cos(x)$, then then $du = -\sin(x)dx$

$$\int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C$$

H. Evaluate $\int 3x(3x-8)^{11}dx$ let $u = 3x-8$, then $du = 3dx$ and $x = \frac{u+8}{3}$

$$\int 3x(3x-8)^{11}dx = \int x(3x-8)^{11}(3dx) = \int\left(\frac{u+8}{3}\right)u^{11}du = \int\left(\frac{u^{12}}{3} + \frac{8u^{11}}{3}\right)du$$

$$= \frac{1}{3}\left(\frac{u^{13}}{13}\right) + \frac{1}{3}\left(\frac{8u^{12}}{12}\right) + C = \frac{(3x-8)^{13}}{39} + \frac{2(3x-8)^{12}}{9} + C$$

IV. Using substitution to find definite integrals: Examples

A. Evaluate $\int_{\pi/6}^{\pi/4} \cos^3(x)dx = \int_{\pi/6}^{\pi/4} [\cos^2(x)]\cos(x)dx = \int_{\pi/6}^{\pi/4} [1-\sin^2(x)][\cos(x)dx]$

1. Method 1: omit limits of integration until integration and back substitution are completed

   a. Let $u = \sin x$, then $du = \cos(x)\,dx$

   b. $\int (1-u^2)du = u - \frac{u^3}{3} = \sin(x) - \frac{\sin^3 x}{3}$

c. Evaluate using original limits of integration:

$$= \sin\left(\frac{\pi}{4}\right) - \frac{1}{3}\sin^3\left(\frac{\pi}{4}\right) - \left[\sin\left(\frac{\pi}{6}\right) - \frac{1}{3}\sin^3\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - \left[\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3\right] = \frac{5\sqrt{2}}{12} - \frac{11}{24}$$

2. Method 2: convert limits of integration at the time of "u" substitution

a. Let u = sin(x), then du = cos(x) dx
b. Conversion of limits of integration:

$$x = \frac{\pi}{4} \rightarrow u = \frac{\sqrt{2}}{2} \text{ and } x = \frac{\pi}{6} \rightarrow u = \frac{1}{2}$$

c. $\int_{\pi/6}^{\pi/4} [1 - \sin^2(x)][\cos(x)dx] = \int_{1/2}^{\sqrt{2}/2} (1 - u^2)du$

$$= \left[u - \frac{u^3}{3}\right]_{1/2}^{\sqrt{2}/2} = \frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 - \left[\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3\right] = \frac{5\sqrt{2}}{12} - \frac{11}{24}$$

B. Evaluate $\int_1^e \frac{9\ln x}{x} dx$ let u = ln x, then du = $\frac{dx}{x}$

$$\int \frac{9\ln x}{x} dx = 9\int \ln|x| \frac{dx}{x} = 9\int u\,du = 9\left[\frac{u^2}{2}\right] = 9\left[\frac{(\ln x)^2}{2}\right]_1^e = 9\left(\frac{1}{2} - 0\right) = \frac{9}{2}$$

C. Evaluate $\int_{-2}^{2} \left(\frac{e^x + 1}{e^x}\right)dx = \int_{-2}^{2} (1 + e^{-x})dx$ let u = -x, then du = -xdx

$$\int_{-2}^{2} (1 + e^{-x})dx = -\int (1 + e^u)du = -(u + e^u) = \left[x - e^{-x}\right]_{-2}^{2}$$

$$= 2 - e^{-2} - (-2 - e^2) = 4 - e^{-2} + e^2$$

IV. Integrals of symmetric functions

Suppose f is continuous on $[-a, a]$

A. If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx - 2\int_0^a f(x)dx$

Example: $\int_{-3}^{3} (-x^4 + 3)dx = 2\int_0^3 (-x^4 + 3)dx$

$$= 2\left[-\frac{x^5}{5} + x\right]_0^2 = 2\left(-\frac{243}{5} + 3\right) = -\frac{456}{5}$$

B. If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)dx = 0$: Example: $\int_{-\pi}^{\pi} \sin(x)dx = 0$