

## INTEGRALS

### Section 5.7 Integration Using Trigonometric Substitution

I. **Pythagorean trig identity:**  $\sin^2 x + \cos^2 x = 1$

From this identity we can quickly derive the other two related identities

$$A. \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 1 + \cot^2 x = \csc^2 x$$

$$B. \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$$

II. Evaluate  $\int \cos^3 x dx$

A. Neither substitution nor integration by parts will work here because a factor of  $(\sin x)$  would be introduced.

$$B. \int \cos^3 x dx = \int (\cos^2 x \cdot \cos x) dx = \int (1 - \sin^2 x) \cdot \cos x dx$$

$$\int (1 - \sin^2 x) \cdot \cos x dx = \int (1 - u^2) du \quad \text{let } u = \sin x \Rightarrow du = \cos x$$

$$\int (1 - u^2) du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C$$

C. General rule: Try to rewrite integrand involving powers of sine and/or cosine in a form which has a single sine factor or a single cosine factor. [see next section if integrand has only even powers of sine and/or cosine.]

III. Evaluate  $\int \cos^4 x dx$

A. If integrand contains only even powers of sine and/or cosine, use the following **half-angle identities**

$$1. \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$2. \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$B. \int \cos^4 x dx = \int (\cos^2 x)^2 dx = \int \left[ \frac{1}{2}(1 + \cos 2x) \right]^2 dx = \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4} x + \frac{1}{4} \sin 2x + \frac{1}{4} \int \frac{1}{2}(1 + \cos 4x) dx = \frac{1}{4} x + \frac{1}{4} \sin 2x + \frac{1}{8} x + \frac{1}{8} \left( \frac{1}{4} \sin 4x \right) + C$$

$$= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$$

IV. A trig substitution may sometimes be used to get rid of a root sign

A. If you have a factor of  $\sqrt{a^2 - x^2}$ , let  $x = a(\sin \theta)$

B. If you have a factor of  $\sqrt{a^2 + x^2}$ , let  $x = a(\tan \theta)$

C. If you have a factor of  $\sqrt{x^2 - a^2}$ , let  $x = a(\sec \theta)$

V. Evaluate  $\int_{\sqrt{2}}^2 \frac{1}{t^3 \sqrt{t^2 - 1}} dt$

A. Let  $t = a(\sec \theta)$ , where  $0 \leq \theta < \frac{\pi}{2}$  or  $\pi \leq \theta < \frac{3\pi}{2}$ , and  $a = 1$

[We restrict  $\theta$  to these intervals so that  $\tan \theta$  will be non-negative.]

B.  $t = \sec \theta \Rightarrow dt = \sec \theta \tan \theta d\theta$

C. We must convert the limits of integration in this case.

$$1. \quad \sqrt{2} = \sec \theta = \frac{1}{\cos \theta} \Rightarrow \cos \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \quad (\text{domain of } \cos^{-1} \theta \text{ is } [0, \pi])$$

$$2. \quad 2 = \sec \theta = \frac{1}{\cos \theta} \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$3. \quad \int \frac{1}{t^3 \sqrt{t^2 - 1}} dt = \int \frac{1}{(\sec \theta)^3 \sqrt{(\sec \theta)^2 - 1}} \sec \theta \tan \theta d\theta = \int \frac{1}{(\sec \theta)^3 \sqrt{(\tan \theta)^2}} \sec \theta \tan \theta d\theta$$

$$4. \quad \text{Since } \tan \theta \geq 0, \quad \sqrt{(\tan \theta)^2} = |\tan \theta| = \tan \theta$$

$$\int \frac{\sec \theta \tan \theta}{(\sec \theta)^3 \tan \theta} d\theta = \int \frac{1}{(\sec \theta)^2} d\theta = \int (\cos \theta)^2 d\theta = \int \frac{1}{2} (1 + \cos 2\theta) d\theta \quad [\text{double angle}]$$

$$= \frac{1}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_{\pi/4}^{\pi/3} = \frac{1}{2} \left[ \frac{\pi}{3} + \frac{1}{2} \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{4} + \frac{1}{2} (1) \right]$$

$$= \frac{1}{2} \left( \frac{\pi}{12} + \frac{\sqrt{3}}{4} - \frac{1}{2} \right) = \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}$$