INTEGRALS

Section 5.7 Integration of Rational Functions by Partial Fractions

Objective: Evaluate integrals using partial fraction decomposition

I. Any rational function (a ratio of polynomials) can be integrated by expressing it as a sum of simpler fractions

II. Case 1: Degree of numerator is greater than or equal to degree of denominator - perform the long division first

$$\int \frac{x^3 - 3x}{x + 1}dx = \int\left(x^2 - x - 2 + \frac{2}{x + 1}\right)dx = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 2\ln|x + 1| + C$$

III. Case 2: The denominator is a product of distinct linear factors, where no factor is repeated

A. Background from addition of fractions

1. $\dfrac{2}{x - 1} - \dfrac{1}{x + 2} = \dfrac{2(x + 2) - 1(x - 1)}{(x - 1)(x + 2)} = \dfrac{x + 5}{x^2 + x - 2}$
2. How do we undo this process? [**painfully!**]

B. Integrate $\int \dfrac{x + 5}{x^2 + x - 2}dx$ using partial fraction decomposition

1. $\dfrac{x + 5}{x^2 + x - 2} = \dfrac{x + 5}{(x - 1)(x + 2)} = \dfrac{A}{x - 1} + \dfrac{B}{x + 2}$
2. $x + 5 = A(x + 2) + B(x - 1)$ Multiply BS by LCD

   $x + 5 = Ax + 2A + Bx - B$

   $x = Ax + Bx$ and $5 = 2A - B$

   $\begin{cases} 1 = A + B \\ 5 = 2A - B \end{cases} \Rightarrow A = 2,\ B = -1$ Equate coefficients of like powers
3. $\therefore \int \dfrac{x + 5}{x^2 + x - 2}dx = \int\left(\dfrac{2}{x - 1} - \dfrac{1}{x + 2}\right)dx = 2\ln|x - 1| - \ln|x + 2| + C$

C. Example: $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$

1. $\int \frac{x^2 + 2x - 1}{x(2x - 1)(x + 2)} dx = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2}$
2. $x^2 + 2x - 1 = A(2x - 1)(x + 2) + Bx(x + 2) + Cx(2x - 1)$ Multiply BS by LCD

   $x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$

3. $\begin{cases} 2A + B + 2C = 1 \\ 3A + 2B - C = 2 \\ -2A = -1 \end{cases} \Rightarrow A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$ Equate coefficients of like powers
4. $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx = \int \left(\frac{1}{2x} + \frac{1}{5(2x - 1)} - \frac{1}{10(x + 2)} \right) dx$

   $= \frac{1}{2} \ln |x| + \frac{1}{10} \ln |2x - 1| - \frac{1}{10} \ln |x + 2| + C$

V. Case 3: The denominator is a product of linear factors, some of which are repeated

A. Example: $\int \left(\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \right) dx = \int \left(x + 1 + \frac{4x}{x^3 - x^2 - x + 1} \right) dx$

1. Let Q(x) = $x^3 - x^2 - x + 1$. Then Q(1) = 0 implies that $(x - 1)$ is a factor of of Q(x) = $x^3 - x^2 - x + 1$. Dividing yields $x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1)$.
2. Partial fraction decomposition

   $$\frac{4x}{x^3 - x^2 - x + 1} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 1}$$

3. $4x = A(x - 1)(x + 1) + B(x + 1) + C(x - 1)^2$

   $4x = (A + C)x^2 + (B - 2C)x + (-A + B + C)$

   $A + C = 0 \Rightarrow A = -C$

   $B - 2C = 4 \Rightarrow B = 4 + 2C$

   $-A + B + C = 0$ $C + 4 + 2C + C = 0 \Rightarrow C = -1$

   Then A = 1 and B = 2

4. $\int\left(\dfrac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\right)dx = \int\left(x + 1 + \dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} - \dfrac{1}{x+1}\right)dx$

$= \dfrac{x^2}{2} + x + \ln|x-1| - \dfrac{2}{x-1} - \ln|x+1| + C$

$= \dfrac{x^2}{2} + x + \ln\left|\dfrac{x-1}{x = 1}\right| - \dfrac{2}{x-1} - \ln|x+1| + C$

B. Background: $\dfrac{8}{4x+3} + \dfrac{1}{x^3} - \dfrac{2}{x} = \dfrac{-6x^2 + 4x + 3}{x^3(4x+3)}$

Example: $\int \dfrac{6}{x^3(4x+3)}dx$

$$\frac{6}{x^3(4x+3)} = \frac{A}{4x+3} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}$$

V. Case 4: The denominator contains irreducible quadratic factors, none of which is repeated

A. Evaluate $\int\left(\dfrac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\right)dx$ [degree of numerator $\geq$ degree of denominator]

1. $\int\left(\dfrac{4x^2 - 3x + 2}{4x^2 - 4x + 3}\right)dx = \int\left(1 + \dfrac{x-1}{4x^2 - 4x + 3}\right)dx$
2. $4x^2 - 4x + 3$ is irreducible because its discriminant is negative; therefore it cannot be factored.
3. Complete the square of the polynomial

$$4x^2 - 4x + 3 = 4\left(x^2 - x + \frac{1}{4}\right) + 3 - 1 = 2^2\left(x - \frac{1}{2}\right)^2 + 2 = (2x-1)^2 + 2$$

4. Let u = 2x – 1; then du = 2dx, and $x = \dfrac{u+1}{2}$

B. $\int\left(1 + \dfrac{x-1}{4x^2 - 4x + 3}\right)dx = x + \dfrac{1}{2}\int\left(\dfrac{\frac{1}{2}(u+1) - 1}{u^2 + 2}\right)du$

$= x + \dfrac{1}{4}\int\left(\dfrac{u-1}{u^2+2}\right)du = x + \dfrac{1}{4}\int\left(\dfrac{u}{u^2+2}\right)du - \dfrac{1}{4}\int\left(\dfrac{1}{u^2+2}\right)du$

$$= x + \frac{1}{8}\ln(u^2 + 2) - \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$ table: #17 with $a = \sqrt{2}$

$$= \frac{1}{8}\ln(4x^2 - 4x + 3) - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$$

VI. Case 5: The denominator contains a repeated irreducible quadratic factor

A. Evaluate $\int\left(\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\right)dx$

1. $\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}$

2. $1 - x + 2x^2 - x^3 = A(x^2+1)^2(Bx + C)x(x^2+1) + (Dx + E)x$

   Solution: $A = 1$, $B = -1$, $C = -1$, $D = 1$, and $E = 0$

B. $\int\left(\frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\right)dx = \int\left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$

$$= \int\frac{dx}{x} - \int\frac{x}{x^2+1}dx - \int\frac{dx}{x^2+1} + \int\frac{x}{(x^2+1)^2}dx$$ let $u = x^2 + 1$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$$