APPLICATIONS OF INTEGRATION

6.5 Applications to Physics and Engineering

Objective: Calculate work and centers of mass

I. Force = push or pull on an object

A. A constant force does not change in magnitude while it is being applied to an object
B. A variable force changes while it is being applied to an object
(e.g. because of friction or inertia)
C. If an object moves along a straight line with position function s(t), then the force F on the object (in the same direction) is defined by Newton's Second Law of Motion as the product of its mass m and its acceleration: $F = m \bullet \frac{d^2s}{dt^2}$

II. ***Work*** = constant force F x distance d moved in direction of the force
A. W = Fd [work = force x distance]
B. U.S. Customary system

1. Force is in lb
2. Displacement is in ft or in
3. Work is in ft-lb or in-lb [1 ft-lb $\approx$ 1.36 J]

C. SI metric system

1. Force is in Newtons (N) [1 N = 1 kg-m/ s^2]
2. Displacement is in m
3. Work is in N-m = joule (J)

III. A 1.2-kg book is lifted off the floor and put on a desk 0.7 m high

A. F = mg = (2.2)(9.8) = 11.76 N
B. W = Fd = (11.76)(0.7) $\approx$ 8.2 J

IV. A 20-lb weight is lifted 6 ft off the ground

A. Force = 20 lb [weight is a force]
B. W = Fd = (20)(6) = 120 ft-lb

V. Example 1 [force is variable in this example]

A. If f is continuous on [a, b] and an object is moved along the x-axis from x = a to x = b, then $W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$

B. A force of $(x^2 + 2x)$ lb acts on a particle located at a distance x ft from the origin. How much work is done in moving it from x = 1 to x = 3?

$$W = \int_1^3 (x^2 + 2x)dx = \left[\frac{x^3}{3} + x^2\right]_1^3 = \frac{50}{3}\text{ ft-lb}$$

C. An analogous definition can be stated for an interval on the y-axis

V. Hooke's Law: The force f(x) required to maintain a spring stretched x units beyond its natural length is given by f(x) = kx, where k is a positive constant called the spring constant.

Example 2: A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. Find the work done in stretching the spring from 15 cm to 18 cm

1. The distance stretched is 5 cm = .05 m
2. f(.05) = 40 $\Rightarrow$ 0.05k = 40 $\Rightarrow$ k = 800
3. ˆ f(x) = 800x
4. Work done in stretching spring from 15 cm to 18 cm is

$$W = \int_{.05}^{.08} 800x\,dx = 800\left[\frac{x^2}{2}\right]_{.05}^{.08} = 1.56\text{J}$$

VI. Suppose we wish to move a solid vertically (e.g. pumping out fluid or raising elevator)

Example 3 on p. 476 [see drawing]

A. Divide the interval [2, 10] into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and choose x_i^* in the ith subinterval

B. Water is sliced into n layers

C. Each slice is approximately a circular cylinder

1. Thickness of slice = Äx [interesting notation!]
2. Radius of slice = r_i and $\frac{r_i}{10 - x_i^*} = \frac{4}{10} \Rightarrow r_i = \frac{2}{5}(10 - x_i^*)$
3. Volume of slice $\approx p r_i^2 \Delta x = \frac{4p}{25}(10 - x_i^*)^2 \Delta x$
4. Mass = density x volume

   $$m_i \approx 1000 \bullet \frac{4p}{25}(10 - x_i^*)^2 \Delta x = 160p(10 - x_i^*)^2 \Delta x$$

5. Force must overcome gravity

   $$F_i = m_i g\ \blacksquare\ (9.8)[160p(10 - x_i^*)^2 \Delta x] \approx 1570p(10 - x_i^*)^2 \Delta x$$

6. Work to raise one slice

$W_i \approx F_i x_i^* \approx 1570p\, x_i^*(10 - x_i^*)^2 \Delta x$

7. $W = \lim_{n \to \infty} \sum_{i=1}^{n} 1570p\, x_i^*(10 - x_i^*)^2 \Delta x$

$= \int_2^{10} 1570p\, x(10 - x)^2 dx = 1570p \int_2^{10} (100x - 20x^2 + x^3) dx \approx 3.4 \times 10^6 \text{ J}$

VII. Moments and centers of mass

A. Goal is to find point P on which a thin plate of any given shape balances horizontally [center of mass or center of gravity]
B. Weight is determined by gravity
[weight on moon = 1/6 weight on Earth]
C. Mass is constant
D. F = ma [Newton's second law of motion]

1. Slug = unit of mass in U.S. Customary system
[$a = 32 \text{ ft/sec}^2$]
2. Kilogram = unit of mass in SI metric system
[$a = 9.81 \text{ m/sec}^2$]

D. We assume that a mass is concentrated at a point [a point-mass]

1. Let two masses be attached to a rod of negligible mass on opposite sides of the fulcrum [balance point]

a. Point-masses m_1 and m_2 are placed at distances x_1 and x_2 from the fulcrum P
b. The rod balances if $m_1 x_1 = m_2 x_2$

i. Law of the Lever
ii. $m_1 x_1$ is the moment of mass m_1 with respect to the origin
iii. $m_2 x_2$ is the moment of mass m_2 with respect to the origin

2. Suppose the rod lies along the x-axis

Let x-coordinates of m_1, m_2, and P be x_1, x_2, and $\overline{X}$

i. $m_1(\overline{X} - x_1) = m_2(\overline{X} - x_2)$
ii. $\overline{X} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

E. Center of mass $\overline{X}$ is obtained by adding the moments of the masses and dividing by the total mass $m = m_1 + m_2 \Rightarrow \overline{X} = \dfrac{\sum_{i=1}^{n} m_i x_i}{m}$

VIII. If S is a system of point-masses in a coordinate plane

A. Moment of m_1 about x-axis = $m_1 y_i$

Moment of S about x-axis = $M_x = \sum_{i=1}^{n} m_i y_i$

B. Moment of m_1 about y-axis = $m_1 x_i$

Moment of S about y-axis = $M_y = \sum_{i=1}^{n} m_i x_i$

E. Center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$

IX. Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at points (-1, 1), (2, -1), and (3, 2), respectively

A. $M_y = 3(-1) + 4(2) + 8(3) = 29$

B. $M_x = 3(1) + 4(-1) + 8(2) = 15$

C. $m = 3 + 4 + 8 = 15$

D. $\overline{x} = \dfrac{M_y}{m} = \dfrac{29}{15}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{15}{15}$

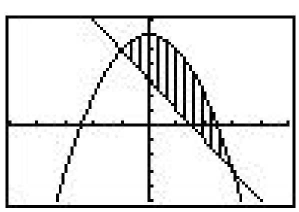
E. Center of mass = $\left(\dfrac{29}{15}, 1\right)$

X. Find the centroid of a lamina (flat plate) with uniform density ñ

A. Let the region be bounded by the graphs of $y = 6 - x^2$ and $y = 3 - 2x$

1. For simplicity let $r = 1$ [it would cancel out in the calculations anyway!]

2. Draw graph of region



3. Draw a typical rectangle and label its center C

   a. Area of rectangle = length x width = $[(6-x^2)-(3-2x)]dx$

   b. Distance from x-axis to C = $\frac{1}{2}[(6-x^2)+(3-2x)]$

   c. Moment about x-axis = $\frac{1}{2}[(6-x^2)+(3-2x)]\,[(6-x^2)-(3-2x)]dx$

   d. Distance from y-axis to C = x

   e. Moment about y-axis = $x[(6-x^2)-(3-2x)]dx$

B. $M_x = \int_{-1}^{3}\frac{1}{2}\,[(6-x^2)+(3-2x)]\,[(6-x^2)-(3-2x)]dx$

$$= \frac{1}{2}\int_{-1}^{3}[(6-x^2)^2-(3-2x)^2]dx = \frac{1}{2}\int_{-1}^{3}(x^4-16x^2+12x+27)dx = \frac{416}{15}$$

C. $M_y = \int_{-1}^{3}x\,[(6-x^2)-(3-2x)]dx = \frac{1}{2}\int_{-1}^{3}(3x+2x^2-x^3)dx = \frac{32}{3}$

D. $m = A = \int_{-1}^{3}[(6-x^2)-(3-2x)]dx = \frac{1}{2}\int_{-1}^{3}(3+2x-x^2)dx = \frac{32}{3}$

1. $\bar{x} = \frac{M_y}{m} = \frac{\left(\frac{32}{3}\right)}{\left(\frac{32}{3}\right)} = 1$

2. $\bar{y} = \frac{Mx}{m} = \frac{\left(\frac{416}{15}\right)}{\left(\frac{32}{3}\right)} = \frac{13}{5}$

E. Centroid = $\left(1, \frac{13}{5}\right)$