INFINITE SEQUENCES AND SERIES

8.1 Sequences

Objective: Determine if a sequence converges or diverges

I. A sequence is a list of numbers written in a definite order

   A. We are dealing only with ***infinite*** sequences
   B. $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ may also be written as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$
   C. n is always a positive integer

II. Some sequences can be defined by a formula for the nth term

   A. $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ [not all sequences begin with n = 1!]
   B. $a_n = \sqrt{n-3},\ n \geq 3$
   C. $\{0, 1, \sqrt{2}, \sqrt{3}, 2, \ldots, \sqrt{n-3}, \ldots\}$

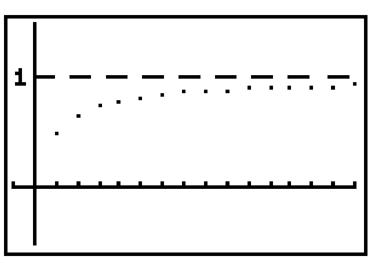
III. Some sequences cannot be defined by a formula

   A. See Example 2 on p. 560
   B. Fibonacci sequence $\{f_n\}$ is defined recursively

      1. $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2},\ \ n \geq 3$
      2. 1, 1, 2, 3, 5, 8, 13, 21, . . .

IV. Consider the sequence $a_n = \dfrac{n}{n+1}$

   A. The terms are $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots$

   B. 

   Set calculator `MODE` to `Seq`

   $\lim\limits_{n \to \infty} \dfrac{n}{n+1} = 1$ graphically

   The sequence ***seems*** to converge to 1

V. $\lim\limits_{n \to \infty} a_n = L$ if we can make the terms a_n as close to L as we like by taking n sufficiently large

A. If $\lim_{n\to\infty} a_n$ exists [as a real number], then the sequence converges

B. If $\lim_{n\to\infty} a_n$ does not exist, then the sequence diverges

C. If a_n becomes large as n becomes large, we write $\lim_{n\to\infty} a_n = \infty$

[the sequence ***diverges*** to ∞]

VI. Limit laws for convergent sequences
[compare to limit laws for functions in Section 2.3]

A. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim_{n\to\infty} c = c$
2. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
4. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
5. $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
6. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ $\qquad \lim_{n\to\infty} b_n \neq 0$

VII. Other limit theorems

A. If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

1. Find $\lim_{n\to\infty} \frac{\ln n}{n}$

   We note that the numerator and denominator both approach infinity, ***however*** l'Hospital's Rule only applies to ***functions*** of a real variable

2. Let $f(x) = \frac{\ln x}{x}$ [related function]
3. Then $\lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$
4. Therefore, $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

B. $\lim_{n\to\infty} \frac{1}{n^r} = 0$, if $r > 0$

Application of this theorem to a previous problem:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+0} = 1$$ [r = 1 in this problem]

C. Squeeze theorem for sequences:
If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

1. Does $a_n = \frac{n!}{n^n}$ converge or diverge?
2. The numerator and denominator both approach infinity as n becomes large, but there is no corresponding function, so we cannot use l'Hospital's Rule.
3. $a_1 = 1,\ a_2 = \frac{1 \cdot 2}{2 \cdot 2},\ a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$, and $a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$.
4. $a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right) \Rightarrow 0 < a_n \le \frac{1}{n}$.
5. Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, the Squeeze Theorem proves that $\lim_{n\to\infty} a_n = 0$.
6. Therefore, the sequence $\{a_n\}$ converges to 0.

D. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

1. Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ $\left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots\right\}$
2. $\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$
3. Therefore, $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

VIII. The sequence $\{r^n\}$ is convergent if $-1 < r \le 1$ and divergent for all other values of r

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

IX. A sequence is ***monotonic*** if it is either [strictly] increasing or [strictly] decreasing

A. A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \ge 1$.
B. A sequence $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \ge 1$.
C. Is $\left\{\frac{n-2}{n+2}\right\}$ increasing or decreasing?

1. $\left\{-\frac{1}{3}, 0, \frac{1}{5}, \frac{1}{3}, \frac{3}{7}, \ldots\right\}$ ***appears*** to be increasing

2. If $\left\{\frac{n-2}{n+2}\right\}$ is increasing, then $\left\{\frac{n-2}{n+2}\right\} < \left\{\frac{(n+1)-2}{(n+1)+2}\right\}$,

   i.e. $\left\{\frac{n-2}{n+2}\right\} < \left\{\frac{n-1}{n+3}\right\}$

3. This implies that (n – 2)(n + 3) < (n + 2)(n – 1), since both denominators > 0.
   $n^2 + n - 6 < n^2 + n - 2 \Rightarrow -6 < -2$

4. Since $-6 < -2$ for all $n \geq 1$, the sequence is increasing, and therefore monotonic.

D. Is $a_n = \frac{n}{n^2+1}$ increasing or decreasing?

1. $\left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \ldots\right\}$ appears to be decreasing

2. If $\left\{\frac{n}{n^2+1}\right\}$ is decreasing, then $\frac{n}{n^2+1} > \frac{n+1}{(n+1)^2+1}$.

3. This implies that $n[(n+1)^2+1] > (n+1)(n^2+1)$ since both denominators > 0.

4. $n^3 + 2n^2 + 2n > n^3 + n^2 + n + 1 \Rightarrow n^2 + n > 1$, and $n^2 + n > 1$ is always true since $n \geq 1$.

5. Therefore, $a_n > a_{n+1}$ for all $n \geq 1$, which means that $\{a_n\}$ is decreasing and monotonic.

E. Is $\left\{(-1)^n \frac{n+1}{n}\right\}$ increasing or decreasing?

a. $\left\{-2, \frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, \ldots\right\}$

b. Since $-2 < \frac{3}{2}$ but $\frac{3}{2} > -\frac{4}{3}$, $\left\{(-1)^n \frac{n+1}{n}\right\}$ is neither increasing nor decreasing, and is therefore not monotonic.

F. Is $a_n = \ln(n+1) - \ln n$ increasing or decreasing?

a. { 0.693, 0.405, 0.288, 0.223, . . . } appears to be decreasing

b. Let $f(x) = \ln(x+1) - \ln x = \ln\left(\frac{x+1}{x}\right)$

c. $f'(x) = \frac{\left[\frac{x(1)-(x+1)(1)}{x^2}\right]}{\left[\frac{x+1}{x}\right]} = \frac{-1}{x(x+1)} < 0$ for all $n \geq 1$

d. Since f(x) is a decreasing function, $\{a_n\}$ is a decreasing sequence, and is therefore monotonic.

X. A sequence $\{a_n\}$ is a ***bounded sequence*** if it is bounded above and below.

A. A sequence $\{a_n\}$ is ***bounded above*** if there is a number M such that $a_n \le M$ for all $n \ge 1$.

B. A sequence $\{a_n\}$ is ***bounded below*** if there if a number m such that $m \le a_n$ for all $n \ge 1$.

C. The sequence $a_n = n$ is bounded below $[a_n > 0]$, but not above.

D. The sequence $a_n = \frac{n}{n+1}$ is bounded since $0 < a_n < 1$ for all $n \ge 1$.

XI. Every bounded, monotonic sequence is convergent.