INFINITE SEQUENCES AND SERIES

8.5 Power Series

Objective: Determine if a power series converges or diverges

I. Power series: $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

A. A power series may converge for some values of x and diverge for others
B. The sum of the series is a function

1. $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$
2. Domain is set of all x for which series converges
3. f(x) is similar to a polynomial, but has infinitely many terms

C. If $c_n = 1$ for all n, the power series becomes the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

1. The series converges if $|x| < 1$; the interval of convergence is $(-1, 1)$.
2. Radius of convergence is 1.

II. $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n + \ldots$

A. Power series centered at a.
B. $(x-a)^0 = 1$ even if x = a. [for series]
C. Always converges if x = a.

III. Example 1: For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

A. Ratio test: If $x \neq 0$, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| =$

$|x| \lim_{n\to\infty}(n+1) = \infty$

B. Thus , the series diverges when $x \neq 0$.
C. If x = 0, all terms are zero and the series converges.
D. Interval of convergence is {0} . [A collapsed interval]
E. Radius of convergence R = 0.

IV. Example 2 : For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

A. Use the ratio test to show that the series is absolutely convergent [and therefore] convergent when $|x - 3| < 1$.

B. $-1 < x - 3 < 1 \Rightarrow 2 < x < 4 \Rightarrow$ the open interval of convergence is (2, 4).

C. Radius of convergence R = 1

D. Must test endpoints x = 2 and x = 4 with some other test since ratio test fails at endpoints.

1. When x = 4, the series become the harmonic series which is divergent.
2. When x = 2, the series converges by the Alternating Series Test.

E. Interval of convergence is [2, 4)

V. Example 3: Bessel function of order 0: $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

A. Use the Ratio Test to show that the series converges for all x.

B. Interval of convergence is $(-\infty, \infty) \Rightarrow$ domain of the Bessel function is $(-\infty, \infty)$.

C. Radius of convergence R = ∞.

VI. For a given power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ there are 3 possibilities:

A. The series converges only when x = a. [See Example 1]

B. The series converges for all x. [See Example 3]

C. There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$. [See Example 2]

VII. Find radius of convergence and interval of convergence of $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

A. The Riot Test can usually be used to find the open interval of convergence.

B. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \bullet \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = 3\,|x|\lim_{n\to\infty}\sqrt{\frac{1+1/n}{1+2/n}} = 3\,|x|$

C. By ratio test: series converges if $3\,|x| < 1$ and diverges if $3\,|x| > 1 \Rightarrow$ series converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$

1. Radius of convergence is R = $\frac{1}{3}$
2. Series converges in $\left(-\frac{1}{3}, \frac{1}{3}\right)$, but what about the endpoints?

C. If x = $-\frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots$

1. $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

2. Divergent p-series with $p = \frac{1}{2} < 1$

D. If $x = \frac{1}{3}$, $\sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which converges by Alternating Series Test.

E. Interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right]$.