

## INFINITE SEQUENCES AND SERIES

### 8.6 Representations of Functions as Power Series

Objective: Express a known function as the sum of infinitely many terms

I. This is a useful procedure for

- A. Integrating functions that don't have elementary antiderivatives.
- B. Solving differential equations. [Used in Calculus IV]
- C. Approximating functions by polynomials.

II. Recall  $\frac{1}{x-1} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$

III. Example 1: Express  $\frac{1}{1+x^2}$  as the sum of a power series and find the interval of convergence.

A. Replace  $x$  by  $-x^2$  in previous equation

B.  $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$

1. This is a geometric series; converges when  $|-x^2| < 1 \Rightarrow |x^2| < 1 \Rightarrow |x| < 1$ .
2.  $|x| < 1 \Rightarrow$  interval of convergence is  $(-1, 1)$

IV. Differentiation and integration of power series [term-by-term]

A. The sum of a power series is a function  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ .

1. Domain is the interval of convergence of the series.
2. Can treat as a polynomial.

B. If the power series  $\sum_{n=0}^{\infty} c_n(x-a)^n$  has radius of convergence  $R > 0$ , then the

function  $f$  defined by  $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$

$= \sum_{n=0}^{\infty} c_n(x-a)^n$  is differentiable (and therefore continuous) on the interval

$(a-R, a+R)$ .

1.  $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1}$ .

2.  $\int f(x) dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \dots = C + \sum_{n=1}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ .

3. The radius of convergence of the derivative and integral are both R.
4. The interval of convergence may change [only at an endpoint].
5. This method only works with power series.

V. Find a power series representation for  $\frac{1}{(1-x)^2}$ .

A. Begin with  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$ ,  $|x| < 1$

B. Differentiate each side to obtain  $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots = \sum_{n=0}^{\infty} (n+1)x^n$   $|x| < 1$ .

1.  $\sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} (n+1)x^n$  [replacing n with n + 1]

2. R still equals 1

VI. Evaluate  $\int \frac{1}{(1-x)^7} dx$  as a power series

A.  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$

1. Replace x with  $-x^7$

2.  $\frac{1}{(1-x)^7} = \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n = \sum_{n=0}^{\infty} (-1)^n x^{7n} = 1 - x^7 + x^{14} - \dots$

B.  $\int \frac{1}{(1-x)^7} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n} dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} = C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \dots$

C. The series converges for  $| -x^7 | < 1$ , i.e. for  $|x| < 1$ .

D. See Example 8:  $\int_0^{0.5} \frac{1}{(1-x)^7} dx = \left[ x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \dots \right]_0^{0.5} \approx 0.49951374$

Using alternating series estimation theorem with  $n = 4$  gives the answer correct to within  $10^{-7}$ .