Introduction to Limits

- The concept of a limit is our doorway to calculus. This lecture will explain what the limit of a function is and how we can find such a limit. Be sure you understand function notation at this point, it will be used throughout the remainder of the course.

- Consider the function

$$f(x) = \frac{x^4 - 16}{x - 2}$$

Note that the domain of $f$ is $\{x | x \neq 2\}$. What does the graph look like near $x = 2$? We can certainly graph the function with our graphing calculator & see what happens. Before we do this, though, let’s look at the value (output) of $f$ for values of $x$ close to 2. These can be seen in the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1.99</th>
<th>1.999</th>
<th>1.9999</th>
<th>2.0001</th>
<th>2.001</th>
<th>2.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>31.761</td>
<td>31.976</td>
<td>31.998</td>
<td>32.002</td>
<td>32.024</td>
<td>32.241</td>
</tr>
</tbody>
</table>

Note that as $x$ approaches (gets close to) 2, the value of $f(x)$ seems to be approaching 32. We say "The limit of $f(x)$ as $x$ approaches 2 is 32" and write

$$\lim_{x \to 2} f(x) = 32$$

Graph the function over the intervals $0 \leq x \leq 4$ and $0 \leq y \leq 40$. Describe what is happening to the graph of $f$ as $x$ approaches 2. Note how the graph shows the same behavior as the table above describes.

- The following is an intuitive definition for the limit of a function: If $f(x)$ gets arbitrarily close to a real number $L$ as $x$ approaches (gets close to) $a$, then

$$\lim_{x \to a} f(x) = L \text{ OR } f(x) \to L \text{ as } x \to a$$

We say this as "the limit of $f(x)$ as $x$ approaches $a$ is $L".  

- The phrase "gets arbitrarily close to" basically means as close as we like.
- If $f(x)$ does NOT get arbitrarily close to a real number $L$, we say that the limit does not exist. We will write DNE from now on if the limit does not exist.
- Note that in the previous example $f(2)$ does not exist (is undefined), but $\lim_{x \to 2} f(x)$ DOES exist.

Hence, for a limit to exist at $a$, the function does not have to be defined at $a$.

Finding Limits Numerically & Graphically

- When finding limits numerically we will basically construct a table of values as we did in the example above. When finding limits graphically we will look at the graph of the function to estimate limits. Here are some examples:

1. Estimate numerically $\lim_{x \to 9} g(x)$ if

$$g(x) = \frac{\sqrt{x} - 3}{x - 9}$$

We construct a table of values for $g(x)$ for values of $x$ close to 9.

<table>
<thead>
<tr>
<th>$x$</th>
<th>8.9</th>
<th>8.99</th>
<th>8.999</th>
<th>9.001</th>
<th>9.01</th>
<th>9.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>0.16713</td>
<td>0.16671</td>
<td>0.16667</td>
<td>0.16666</td>
<td>0.16662</td>
<td>0.16621</td>
</tr>
</tbody>
</table>

It appears that as $x$ approaches 9 that $g(x)$ is getting closer to 0.16666... or $0.1\overline{6} = \frac{1}{6}$ (see below).
Hence, it appears that

\[ \lim_{x \to 9} g(x) = \frac{1}{6} \]

Side Note: Do you know how to convert from a non-terminating but repeating decimal expansion like 0. 1 6 6 to its equivalent fraction? Here’s one way:

Let \( n = 0. 1 6 6 \). Then \( 10n = 1. 6 6 \) and \( 100n = 16. 6 6 \). Thus

\[ 90n = 100n - 10n = 16. 6 6 - 1. 6 6 = 15 \]

Thus

\[ 90n = 15 \Rightarrow n = \frac{15}{90} = \frac{1}{6} \]

2. Construct a table of values for \( f(x) = \frac{\sin x}{x} \) for \( x \) close to zero to estimate \( \lim_{x \to 0} \frac{\sin x}{x} \)

What mode should we be in? (radian or degree?)

Here is such a table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>±0.1</th>
<th>±0.01</th>
<th>±0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\sin x}{x} \approx )</td>
<td>0.99833417</td>
<td>0.9999833</td>
<td>0.9999998</td>
</tr>
</tbody>
</table>

Thus it appears that

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1 \]

This is an important limit we will see again. Look at the graph of \( \frac{\sin x}{x} \) over the intervals \(-\pi \leq x \leq \pi \) & \(-2 \leq y \leq 2\) to confirm the numerical approach.

3. An example of a limit that does not exist (DNE).

Consider the function \( f(x) = \sin \frac{1}{x} \). Note that the domain of \( f \) is all real numbers except 0. What can we say about

\[ \lim_{x \to 0} \sin \frac{1}{x} \]

When we try to graph this function for values of \( x \) near zero, our graphing calculator has problems.

Graph \( f \) over the intervals \(-2 \leq y \leq 2\) and (1) \(-3 \leq x \leq 3\), (2) \(-1 \leq x \leq 1\), (3) \(-0.1 \leq x \leq 0.1\), and finally (4) \(-0.01 \leq x \leq 0.01\). What do you observe? What is happening?

Recall that

\[ \sin t = 1 \hspace{0.5cm} \text{if} \hspace{0.5cm} t = \frac{\pi}{2} + 2n\pi = \frac{\pi}{2} + \frac{4n\pi}{2} = \frac{\pi(1 + 4n)}{2} \hspace{0.5cm} \text{for} \hspace{0.5cm} n = 0, \pm 1, \pm 2, \pm 3, \ldots \]

and

\[ \sin t = -1 \hspace{0.5cm} \text{if} \hspace{0.5cm} t = \frac{3\pi}{2} + 2n\pi = \frac{3\pi}{2} + \frac{4n\pi}{2} = \frac{\pi(3 + 4n)}{2} \hspace{0.5cm} \text{for} \hspace{0.5cm} n = 0, \pm 1, \pm 2, \pm 3, \ldots \]

So

\[ \sin \frac{1}{x} = 1 \hspace{0.5cm} \text{when} \hspace{0.5cm} \frac{1}{x} = \frac{\pi(1 + 4n)}{2} \hspace{0.5cm} \text{or when} \hspace{0.5cm} x = \frac{2}{\pi(1 + 4n)} \hspace{0.5cm} \text{for} \hspace{0.5cm} n = 0, \pm 1, \pm 2, \pm 3, \ldots \]

Thus \( f(x) = 1 \) when

\[ x = \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \frac{2}{13\pi}, \ldots, \frac{2}{\pi(1 + 4n)}, \ldots \]

Note that as \( n \to \infty \), \( x \to 0 \). Or saying it another way, between any positive number \( x \) and zero, there are an INFINITE number of times when \( f(x) = 1 \). In a similar manner,
\[
\sin \frac{1}{x} = -1 \text{ when } \frac{1}{x} = \frac{\pi(3 + 4n)}{2} \text{ or when } x = \frac{2}{\pi(3 + 4n)} \text{ for } n = 0, \pm 1, \pm 2, \pm 3, \ldots
\]

Thus \( f(x) = -1 \) when
\[
x = \frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi}, \frac{2}{15\pi}, \ldots, \frac{2}{\pi(3 + 4n)}, \ldots
\]

Note that as \( n \to \infty, x \to 0 \). Or saying it another way, between any positive number \( x \) and zero, there are an INFINITE number of times when \( f(x) = -1 \).

Thus we see that as \( x \) gets close to zero, \( f(x) \) begins to wildly oscillate between \(-1\) and \(1\). In essence, \( f(x) \) can never "settle down" and approach any one limit. Thus \( \lim_{x \to 0} \sin \frac{1}{x} \) DNE.

4. When Technology Fails.
We saw in the last example that our graphing calculator had troubles graphing the function for values of \( x \) close to zero. This example shows another type error we can run into.

If \( g(t) = \frac{\sqrt{t^4 + 1} - 1}{t^4} \), estimate numerically the following limit
\[
\lim_{t \to 0} g(t)
\]

We proceed by constructing a table of values for \( x \) close to zero

<table>
<thead>
<tr>
<th>( t )</th>
<th>±0.1</th>
<th>±0.01</th>
<th>±0.001</th>
<th>±0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(t) )</td>
<td>0.49999</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Up to \( x = \pm 0.01 \), we may guess that the limit appears to be approaching \( \frac{1}{2} \). But, if we get closer to zero, we see the limit appears to be zero. What is going on?

The problem is that when your graphing calculator evaluates \( \sqrt{t^4 + 1} \) for small values of \( t \), the result is very close to 1. In fact, if you plug in \( t = \pm 0.001 \) in the TI-84 the result is zero. This is due to the limitations of the graphing calculator & the number of digits the calculator is able to carry (Graph \( g \) with \( 0 \leq y \leq 1 \) over (1) \(-4 \leq x \leq 4 \) and the (2) \(-0.01 \leq x \leq 0.01 \). The value of this limit is \( \frac{1}{2} \) which will be able to show later.

One-Sided Limits

Consider the function

\[
f(x) = \begin{cases} 
1 & \text{if } x < 2 \\
3 & \text{if } x > 2 
\end{cases}
\]

The graph of \( f \) is shown below

Note that as \( x \) approaches 2 from the left (or from the negative side or from below) \( f(x) \) approaches 1 (it is always 1 for \( x < 2 \)). But as \( x \) approaches 2 from the right (or from the positive side or from above) \( f(x) \) approaches 3. Since we do not approach any ONE value from both "sides", \( \lim_{x \to 2} f(x) \) DNE.

When \( f(x) \) approaches 1 as \( x \) approaches 2 from the left we write
\[
\lim_{x \to 2^-} f(x) = 1
\]
and we say "the limit of \(f(x)\) as \(x\) approaches 2 from the left is 1" or "the left-hand limit of \(f(x)\) as \(x\) approaches 2 is 1". Similarly, When \(f(x)\) approaches 3 as \(x\) approaches 2 from the right we write
\[
\lim_{x \to 2^+} f(x) = 3
\]
and we say "the limit of \(f(x)\) as \(x\) approaches 2 from the right is 3" or "the right-hand limit of \(f(x)\) as \(x\) approaches 2 is 3".

\[\text{With one-sided limits we have the following useful theorem}
\]
\[
\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)
\]
That is, \(\lim_{x \to a} f(x) = L\) if and only if both one-sided limits exist and are both equal to \(L\).

\[\text{Consider the function}
\]
\[
\frac{|x|}{x} = \begin{cases} 
-1 & \text{if } x < 0 \\
1 & \text{if } x > 0 
\end{cases}
\]

The graph of the function is shown below

\[\text{Note that } \lim_{x \to 0^-} \frac{|x|}{x} = -1, \text{ but } \lim_{x \to 0^+} \frac{|x|}{x} = 1. \text{ Both one-sided limits exist, but they are not equal. Thus}\]
\[\lim_{x \to 0} \frac{|x|}{x} \text{ DNE.}\]

\[\text{Consider the function } h(x) \text{ whose graph is shown below. Find the following limits (if they exist)}
\]
\[\text{ (a) } \lim_{x \to -2^-} h(x) \text{ (b) } \lim_{x \to -2^+} h(x) \text{ (c) } \lim_{x \to -2} h(x) \text{ (d) } \lim_{x \to 1^-} h(x) \text{ (e) } \lim_{x \to 1^+} h(x) \text{ (f) } \lim h(x)\]

\[\text{Infinite Limits}
\]
\[\text{Let’s try to find the limit } \lim_{x \to 0} \frac{1}{x^2} \text{ We proceed numerically, constructing a table of values for } x \text{ close to zero.}\]

<table>
<thead>
<tr>
<th>(x)</th>
<th>±0.1</th>
<th>±0.01</th>
<th>±0.001</th>
<th>±0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{1}{x^2})</td>
<td>100</td>
<td>10,000</td>
<td>1,000,000</td>
<td>100,000,000</td>
</tr>
</tbody>
</table>
As \( x \) gets closer to zero, the value of \( \frac{1}{x^2} \) continues to get bigger and bigger. It does NOT approach any finite number. Thus, we approach no FINITE limit. To indicate this kind of behavior we introduce the notation

\[
\lim_{x \to 0} \frac{1}{x^2} = \infty
\]

and say that \( f \) has an infinite limit. Note that this does NOT mean \( \infty \) is a number (which it is not). It simply expresses the idea that the value of the function gets arbitrarily large (as large as we want) as \( x \) gets close to 0. When we see this expression, we say "the limit is infinity" or "the function increases without bound".

If a function \( f \) gets arbitrarily large BUT NEGATIVE as \( x \) approaches \( a \), we write

\[
\lim_{x \to a} f(x) = -\infty
\]

We can say similar statements with one-sided limits.

Examples:
1. For the function \( g \) shown below find the following limits or write DNE.

   \[
   \begin{align*}
   &\lim_{x \to -2} g(x) &\lim_{x \to -2^+} g(x) &\lim_{x \to -2^-} g(x) \\
   &\lim_{x \to 3} g(x) &\lim_{x \to 3^+} g(x) &\lim_{x \to 3^-} g(x)
   \end{align*}
   \]

2. Show that \( \lim_{x \to 0^+} \ln x = -\infty \).