More Practice with the Difference Quotient

One form of the difference quotient for the function $f$ is $\frac{f(a+h) - f(a)}{h}$. This quotient represents the slope of the secant line between the points $(a, f(a))$ and $(a + h, f(a + h))$ on the graph of $f$. It is also the average rate of change of the function $f$ between $a$ and $a + h$. This expression is involved in the definition of the derivative of a function which we will shortly study and is the fundamental tool used in differential calculus. I have listed below three general "types" of functions that you may encounter and shown the simplification of the difference quotient. Notice that before simplifying we cannot let $h = 0$. However, after simplification we can plug in $h = 0$. The value of the simplified expression when $h = 0$ represents the slope of what we call the tangent line, but more on that later.... Review these three "types" carefully and then try the ones below (answers are given).

- **Polynomial Function Example:** $f(x) = 3x^2 - 2x + 1$
  $$\frac{f(a+h) - f(a)}{h} = 3(a^2 + 2ah + h^2) - 2a - 2h + 1 - 3a^2 + 2a - 1$$
  $$= 3a^2 + 6ah + 3h^2 - 2a - 2h + 1 - 3a^2 + 2a - 1$$
  $$= 6ah + 3h^2 - 2$$

- **Rational (Fractional) Function Example:** $f(x) = \frac{2}{x+1}$ (Technique: Simplify the compound fraction)
  $$\frac{f(a+h) - f(a)}{h} = \frac{2}{a+h+1} - \frac{2}{a+1}$$
  $$= \frac{2(a+1) - 2(a+h+1)}{(a+1)(a+h+1)}$$
  $$= \frac{-2h}{(a+h+1)(a+1)}$$
  $$= \frac{-2}{(a+h+1)(a+1)}$$

- **Radical (Root) Function Example:** $f(x) = \sqrt{x-3}$ (Technique: Rationalize the numerator)
  $$\frac{f(a+h) - f(a)}{h} = \frac{\sqrt{a+h-3} - \sqrt{a-3}}{h}$$
  $$= \frac{\sqrt{a+h-3} - \sqrt{a-3}}{h} \cdot \frac{\sqrt{a+h-3} + \sqrt{a-3}}{\sqrt{a+h-3} + \sqrt{a-3}}$$
  $$= \frac{(a+h-3) - (a-3)}{h(\sqrt{a+h-3} + \sqrt{a-3})}$$
  $$= \frac{1}{h(\sqrt{a+h-3} + \sqrt{a-3})}$$

Now try and confirm the simplification of the difference quotient for the following functions.

1. $f(x) = 4$; Answer: 0
2. $g(x) = 3 - 7x$; Answer: -7
3. $f(t) = t^2 + 4$; Answer: $2a + h$
4. $h(x) = 4x - x^2$; Answer: $4 - 2a - h$
5. $g(u) = 3u^2 - 5u - 4$; Answer: $6a + 3h - 5$
6. $p(t) = \frac{5}{t}$; Answer: $-\frac{5}{(a+h)t}$
7. $f(x) = \frac{2}{x - 5}$; Answer: $-\frac{2}{(a+h-5)(a-5)}$
8. $f(s) = \sqrt{s}$; Answer: $\frac{1}{\sqrt{a+h} + \sqrt{a}}$
9. $Q(x) = \sqrt{x + 6}$; Answer: $\frac{1}{\sqrt{a+h} + \sqrt{a} + \sqrt{a+6}}$
10. $f(t) = \frac{1}{\sqrt{t}}$; Answer: $-\frac{1}{\sqrt{a+h} \sqrt{a} \left(\sqrt{a} + \sqrt{a+h}\right)}$ (combine "rational" and "radical" techniques)