Applications of Linear Systems and $LU$-Decomposition

Curve Fitting & Polynomial Interpolation

**Theorem:** Given any $n$ points in the $xy$-plane that have distinct $x$-coordinates, there is a unique polynomial of degree $n - 1$ or less whose graph passes through these points.

This polynomial is called an **interpolating polynomial**.

Suppose we have $n$ points: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ and want to find the degree $n - 1$ interpolating polynomial:

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$$

By "plugging in" our $n$ points into this polynomial, we obtain $n$ equations in $n$ unknowns:

$$a_0 + a_1x_1 + a_2x_1^2 + \cdots + a_{n-1}x_1^{n-1} = y_1$$
$$a_0 + a_1x_2 + a_2x_2^2 + \cdots + a_{n-1}x_2^{n-1} = y_2$$
$$\vdots$$
$$a_0 + a_1x_n + a_2x_n^2 + \cdots + a_{n-1}x_n^{n-1} = y_n$$

Since we will know the values of the $x_i$'s and $y_i$'s, the unknowns are the $a_i$'s. Thus the augmented matrix for this system is

$$\begin{bmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\
1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n
\end{bmatrix}$$

Hence, the interpolating polynomial can be found by reducing this matrix to RREF.
Example: Find the quadratic polynomial passing through the points \((-1, 0), (1, 10), \text{ and } (2, 5)\)

Graph of \(y = \frac{2x}{3} + 5x - \frac{10}{3}x^2\)
Network Analysis

Networks are composed of branches and nodes (or junctions) through which something "flows". They are used to model such diverse things as electrical networks, traffic flow, drainage and water supply systems, and production and consumption processes. For example, the branches may be roads, the flow is vehicular traffic, and the nodes are street intersections. The common concept in all these models is that the total flow (of electrical current, traffic, water, manufactured goods, etc.) into any node is equal to the flow out of the node (this is called flow conservation). For example, in the diagram below, the circle represents a node. This node has 35 units flowing out of it and \( x_1 + x_2 \) units flowing into it.

\[
\begin{align*}
&x_1 \\
\downarrow &\quad 35 \\
&x_2
\end{align*}
\]

Thus, we must have

\[ x_1 + x_2 = 35 \]

Since, the flow through a node can be represented by a linear equation (as shown above), a network composed of more than one node can be described by a system of linear equations. Solving the system will yield the flow in each branch of the network.

**Example:** Set up and solve a system of linear equations for the water pipe network shown below. What will all the flow rates be if the water flow is shut off between nodes 5 and 1?
**LU-Decompositions**

A factorization of an $n \times n$ square matrix $A$ as $A = LU$ where $L$ is an $n \times n$ lower triangular matrix and $U$ is an $n \times n$ upper triangular matrix, is called an **LU-decomposition** (or **LU-factorization**) of $A$.

Before showing how such a factorization can be obtained, we will illustrate how such decompositions can be used to solve linear systems.

To solve the linear system $A \vec{x} = \vec{b}$ where $A$ is an $n \times n$ matrix follow these steps:

**Step 1:** Rewrite the system $A \vec{x} = \vec{b}$ as $LU \vec{x} = \vec{b}$

**Step 2:** Define a new matrix $\vec{y}$ by $U \vec{x} = \vec{y}$

**Step 3:** Now solve the system $L \vec{y} = \vec{b}$ for $\vec{y}$

**Step 4:** Finally you can solve the system $U \vec{x} = \vec{y}$ for $\vec{x}$

**Example:** The matrix $A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$ can be shown to have the following **LU-decomposition**:

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Use this to solve the linear system

$$A \vec{x} = \vec{b}$$

**Step 1:** Rewrite the system $A \vec{x} = \vec{b}$ as $LU \vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$
Step 2: Define $y_1$, $y_2$, and $y_3$ to form the new matrix $\bar{y}$ by $U \bar{x} = \bar{y}$

$$
\begin{pmatrix}
1 & 3 & 1 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
$$

Step 3: This yields the system $L \bar{y} = \bar{b}$

$$
\begin{pmatrix}
2 & 0 & 0 \\
-3 & 1 & 0 \\
4 & -3 & 7
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
2 \\
3
\end{pmatrix}
$$

which is the equivalent linear system

\begin{align*}
2y_1 &= 2 \\
-3y_1 + y_2 &= 2 \\
4y_1 - 3y_2 + 7y_3 &= 3
\end{align*}

This system can be easily solved using a process similar to back substitution, except we solve the equations from the top down. This is called **forward substitution** and yields the following solution for $\bar{y}$

\begin{align*}
y_1 &= 1, \\
y_2 &= 5, \\
y_3 &= 2
\end{align*}

Finally, we solve the system $U \bar{x} = \bar{y}$ for $\bar{x}$

$$
\begin{pmatrix}
1 & 3 & 1 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
5 \\
2
\end{pmatrix}
$$

or equivalently:

\begin{align*}
x_1 + 3x_2 + x_3 &= 1 \\
x_2 + 3x_3 &= 5 \\
x_3 &= 2
\end{align*}

which is easy to solve with back substitution:

\begin{align*}
x_1 &= 2, \\
x_2 &= -1, \\
x_3 &= 2
\end{align*}
Credit for popularizing the *LU*-factorization is often given to Alan Turing (1912-1954) who was instrumental in breaking the Nazi’s Enigma codes during WWII as depicted in the 2014 movie *The Imitation Game*.

How do we find a *LU*-decomposition for a given square matrix *A*?

**Theorem:** If *A* is a square matrix that can be reduced to a row echelon form *U* by Gaussian elimination WITHOUT row interchanges, then *A* can be factored as *A* = *LU* where *L* is lower triangular.

We can reduce *A* to REF *U* by multiplying *A* by a sequence of *k* elementary matrices:

\[ E_k \cdots E_2 E_1 A = U \]

Since elementary matrices are invertible, we have

\[ A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U \]

If no row interchanges have been performed, then the above inverses are all lower triangular, so that *L* = *E_1^{-1} E_2^{-1} \cdots E_k^{-1} is lower triangular which yields the desired factorization *A* = *LU*.

**Example:** Find an *LU*-decomposition of

\[
\begin{bmatrix}
  2 & 6 & 2 \\
-3 & -8 & 0 \\
 4 & 9 & 2
\end{bmatrix}
\]

First reduce *A* to REF *U* using Gaussian elimination (without row interchanges):

\[
\begin{align*}
\frac{1}{2}R_1 & \rightarrow R_1 \\
\begin{bmatrix}
  1 & 3 & 1 \\
-3 & -8 & 0 \\
 4 & 9 & 2 \\
\end{bmatrix} & \\
E_1 & = \begin{bmatrix}
  \frac{1}{2} & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \\
\end{bmatrix} & E_1^{-1} & = \begin{bmatrix}
  2 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \\
\end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
R_2 + 3R_1 & \rightarrow R_2 \\
\begin{bmatrix}
  1 & 3 & 1 \\
 0 & 1 & 3 \\
 4 & 9 & 2 \\
\end{bmatrix} & \\
E_2 & = \begin{bmatrix}
  1 & 0 & 0 \\
 3 & 1 & 0 \\
 0 & 0 & 1 \\
\end{bmatrix} & E_2^{-1} & = \begin{bmatrix}
  1 & 0 & 0 \\
 -3 & 1 & 0 \\
 0 & 0 & 1 \\
\end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
R_3 + (-4)R_1 & \rightarrow R_3 \\
\begin{bmatrix}
  1 & 3 & 1 \\
 0 & 1 & 3 \\
 0 & -3 & -2 \\
\end{bmatrix} & \\
E_3 & = \begin{bmatrix}
  1 & 0 & 0 \\
 0 & 1 & 0 \\
 -4 & 0 & 1 \\
\end{bmatrix} & E_3^{-1} & = \begin{bmatrix}
  1 & 0 & 0 \\
 0 & 1 & 0 \\
 4 & 0 & 1 \\
\end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
R_3 + 3R_2 & \rightarrow R_3 \\
\begin{bmatrix}
  1 & 3 & 1 \\
 0 & 1 & 3 \\
 0 & 0 & 7 \\
\end{bmatrix} & \\
E_4 & = \begin{bmatrix}
  1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 3 & 1 \\
\end{bmatrix} & E_4^{-1} & = \begin{bmatrix}
  1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & -3 & 1 \\
\end{bmatrix} \\
\end{align*}
\]

\[
\begin{align*}
\frac{1}{7}R_3 & \rightarrow R_3 \\
\begin{bmatrix}
  1 & 3 & 1 \\
 0 & 1 & 3 \\
 0 & 0 & 1 \\
\end{bmatrix} & \\
E_5 & = \begin{bmatrix}
  1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & \frac{1}{7} \\
\end{bmatrix} & E_5^{-1} & = \begin{bmatrix}
  1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 7 \\
\end{bmatrix} \\
\end{align*}
\]
Thus, $L = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}$

\[
L = \begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 7 \\
\end{bmatrix} = \begin{bmatrix}
2 & 0 & 0 \\
-3 & 1 & 0 \\
4 & -3 & 7 \\
\end{bmatrix}
\]

Hence, the $LU$-decomposition of $A$ is

\[
\begin{bmatrix}
2 & 6 & 2 \\
-3 & -8 & 0 \\
4 & 9 & 2 \\
\end{bmatrix} = \begin{bmatrix}
2 & 0 & 0 \\
-3 & 1 & 0 \\
4 & -3 & 7 \\
\end{bmatrix} \begin{bmatrix}
1 & 3 & 1 \\
0 & 1 & 3 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

*** An easier and quicker method for finding $L$ ***

Look at the nonzero entries of $L$ in the example above and think about how they were obtained

\[
L = \begin{bmatrix}
2 & 0 & 0 \\
-3 & 1 & 0 \\
4 & -3 & 7 \\
\end{bmatrix}
\]

This leads to the follow "bookkeeping" process to construct $L$:

**Step 1:** Reduce $A$ to $U$ via Gaussian elimination without row interchanges. Keep track of multipliers used to introduce leading ones and multipliers used to introduce zeros below the leading ones.

**Step 2:** In each position along the main diagonal of $L$, place the *reciprocal* of the multiplier used to introduce the corresponding leading one in $U$.

**Step 3:** In each position below the main diagonal of $L$, place the *negative* of the multiplier used to introduce the corresponding zero in $U$.

**Step 4:** Form the decomposition $A = LU$. 

Using the bookkeeping method described below, find an $LU$-decomposition for $A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 2 & 0 \\ -4 & -4 & 5 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 2 & 0 \\ -4 & -4 & 5 \end{bmatrix} \quad L = \begin{bmatrix} \cdot & 0 & 0 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot \end{bmatrix}$$