Known convergent/divergent series

- Geometric series: \(\sum_{n=0}^{\infty} ar^n\) will converge if \(|r| < 1\) and converges to the sum \(a/(1-r)\). If \(|r| \geq 1\), the series diverges.

- P-series: The series \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) converges if \(p > 1\) and diverges if \(p \leq 1\). (When \(p = 1\), it is known as the harmonic series.)

- Telescoping series: Split the formula for \(a_n\) using a partial fraction decomposition and notice how terms cancel out for \(S_k\) and evaluate \(\lim_{k \to \infty} S_k\) (these are rare)

**Limit Test for Divergence:**

For the series \(\sum_{n=0}^{\infty} a_n\), if \(\lim_{n \to \infty} a_n\) does NOT equal zero, then \(\sum_{n=0}^{\infty} a_n\) diverges.

Tests for Series with Positive Terms (sections 10.3 and 10.5)

1. **Integral Test (10.3)** – Integrate the series formula as an improper integral and if the integral converges the so does the series, if the integral diverges then so does the series. Works best when the formula for \(a_n\) has an ‘easy’ antiderivative.

2. **Limit Comparison Test (10.3)** – For the series \(\sum_{n=0}^{\infty} a_n\) (the one you’re testing) and \(\sum_{n=0}^{\infty} b_n\) (a series you already know to be convergent or divergent), evaluate the limit \(\lim_{n \to \infty} \frac{a_n}{b_n}\). If the limit is a positive number, then both series behave the same ... this is why you need to already know the convergence/divergence of the series you’re comparing with. Works best when \(a_n\) is a rational expression involving powers of \(n\) so you can compare it to a known p-series.

3. **Direct Comparison Test (10.3)** – For the series \(\sum_{n=0}^{\infty} a_n\) (the one you're testing) you’ll need to come up with an inequality comparison:

   Either \(a_n \leq (a\text{ known convergent series})\) will imply \(\sum_{n=0}^{\infty} a_n\) converges.

   OR \(a_n \geq (a\text{ known divergent series})\) will imply \(\sum_{n=0}^{\infty} a_n\) diverges.

4. **Ratio Test (10.5)** – For the series \(\sum_{n=0}^{\infty} a_n\), evaluate the limit \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n}\) ... if the limit is less than 1 then the series converges, if the limit is greater than 1 then the series diverges, but if the limit equals 1 you can’t make any conclusion and you’ll need to try another test. Works best when you have ‘\(n\)’ being used in exponential and factorial form.

5. **Root Test (10.5)** – For the series \(\sum_{n=0}^{\infty} a_n\), evaluate the limit \(\lim_{n \to \infty} \sqrt[n]{a_n}\) ... if the limit is less than 1 then the series converges, if the limit is greater than 1 then the series diverges, but if the limit equals 1 you can’t make any conclusion and you’ll need to try another test. Works best when you have ‘\(n\)’ being used in exponential form.
Testing Alternating Series (10.4)

These series can exhibit:
1. Absolute Convergence – meaning you disregard the \((-1)^n\) factor and consider the series as an all-positive termed series. This means you can apply one of the tests listed before (integral, limit comparison, ratio, n-th root, etc.) for convergence.
2. Conditional Convergence – meaning it can only converge on the condition that you DO include the \((-1)^n\) factor. This is best tested using the Alternating Series Test – if the terms from \((-1)^n a_n\) are decreasing and their limit is ‘0’ for an alternating series, then it converges. The only way an alternating series can diverge is if the terms for \(a_n\) do NOT limit down to 0. It’s always best to test for absolute convergence first because absolute convergence implies conditional convergence.

Power Series (10.6)

A power series is essentially an infinitely long polynomial taking the form \(\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots\).

The interval of convergence is the set of all ‘\(x\)’ values which can be used in the power series to make it convergent. The most efficient way to determine the interval of convergence is to set up the ratio test for absolute convergence (so ignore any \((-1)^n\) factors you see in the power series formula). DON’T FORGET TO CHECK THE INTERVAL’S ENDPOINTS!

Useful Power Series

<table>
<thead>
<tr>
<th>Function</th>
<th>Power Series Form</th>
<th>Interval of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x) = \frac{1}{1-x})</td>
<td>(f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots)</td>
<td>((-1,1))</td>
</tr>
<tr>
<td>(g(x) = \frac{1}{1+x})</td>
<td>(g(x) = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \ldots)</td>
<td>((-1,1))</td>
</tr>
</tbody>
</table>

Taylor and Maclaurin Series

For any function \(f(x)\) (where \(f(x)\) is usually a function with infinitely many derivatives), the Taylor Series for \(f(x)\) centered at \(x = c\) is defined as ... \(f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n = f(c) + \frac{f'(c)}{1!} (x-c) + \frac{f''(c)}{2!} (x-c)^2 + \frac{f'''(c)}{3!} (x-c)^3 + \ldots\)

If specific type of Taylor Series which uses \(c = 0\) is called a Maclaurin Series. (Maclaurin Series are easier to work with.) \(f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots\)

Handy Maclaurin Series for some ‘popular’ functions:

<table>
<thead>
<tr>
<th>Function and Maclaurin Series form</th>
<th>Interval of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots)</td>
<td>((-\infty, \infty))</td>
</tr>
<tr>
<td>(\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots)</td>
<td>((-\infty, \infty))</td>
</tr>
<tr>
<td>(\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots)</td>
<td>((-\infty, \infty))</td>
</tr>
<tr>
<td>(\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots)</td>
<td>((-1,1))</td>
</tr>
<tr>
<td>(\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 + \ldots)</td>
<td>((-1,1))</td>
</tr>
</tbody>
</table>
| \(\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots\) | \((-1,1)\) |注意到 \(x = 1\) 是包括在内的！