INTEGRATION OF IMPROPER INTEGRALS

Objectives: Evaluate integrals on infinite intervals and intervals which have infinite discontinuities

Improper integrals

- May be defined on an infinite interval
- May have an infinite discontinuity
- Are used in probability distributions

The improper integrals \( \int_{a}^{\infty} f(x) \, dx \) and \( \int_{-\infty}^{b} f(x) \, dx \) are said to be convergent if the corresponding [finite] limit exists and divergent if the [finite] limit does not exist.

Type I: Infinite intervals

- \( A(t) = \int_{1}^{t} \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_{1}^{t} = 1 - \frac{1}{t} \)
- The unbounded region extends indefinitely in a horizontal direction
- Note that \( A(t) < 1 \) no matter how large \( t \) is chosen
- \( \lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left( 1 - \frac{1}{t} \right) = 1 \)
- Area of shaded region approaches 1 as \( t \to \infty \)
- \( \int_{1}^{\infty} \frac{1}{x^2} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x^2} \, dx = 1 \)
- The integral is convergent

Integrate \( \int_{-\infty}^{0} \frac{1}{2x - 5} \, dx \)

HINT: \( \int_{-\infty}^{0} \frac{1}{2x - 5} \, dx = \lim_{t \to \infty} \int_{t}^{0} \frac{1}{2x - 5} \, dx \)
Type 1 improper integral

- If \( \int_a^t f(x) \, dx \) exists for every number \( t \geq a \), then
  \[ \int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx, \]
  provided this [finite] limit exists.

- If \( \int_t^b f(x) \, dx \) exists for every number \( t \leq b \), then
  \[ \int_{-\infty}^b f(x) \, dx = \lim_{t \to -\infty} \int_t^b f(x) \, dx, \]
  provided this [finite] limit exists.

- The improper integrals \( \int_a^\infty f(x) \, dx \) and \( \int_{-\infty}^b f(x) \, dx \) are said to be convergent if the corresponding [finite] limit exists and divergent if the [finite] limit does not exist.

- If both \( \int_a^\infty f(x) \, dx \) and \( \int_{-\infty}^a f(x) \, dx \) are convergent, then we define
  \[ \int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^a f(x) \, dx + \int_a^\infty f(x) \, dx \]
  for any real number \( a \).

Is \( \int_1^\infty \frac{1}{x} \, dx \) convergent or divergent?

- \[ \lim_{t \to \infty} \int_1^t \frac{1}{x} \, dx = \lim_{t \to \infty} \left[ \ln |x| \right]_1^t = \lim_{t \to \infty} (\ln t - \ln 1) = \lim_{t \to \infty} (\ln t) = \infty \]

- Therefore, \( \int_1^\infty \frac{1}{x} \, dx \) is divergent.

- Compare this result to \( \int_1^\infty \frac{1}{x^2} \, dx \) which converges.

- In general, \( \int_1^\infty \frac{1}{x^p} \, dx \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \)

Type 2: Discontinuous integrands

- \( f \) is continuous on a finite interval \([a, b)\)
- The unbounded region is infinite in a vertical direction

Type 2 improper integrals

- If \( f \) is continuous on \([a, b]\) and is discontinuous at \( b \), then
  \[ \int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx, \]
  if this [finite] limit exists.

- If \( f \) is continuous on \([a, b]\) and is discontinuous at \( a \), then
  \[ \int_a^b f(x) \, dx = \lim_{t \to a^+} \int_t^b f(x) \, dx, \]
  if this [finite] limit exists.
• The improper integral \( \int_a^b f(x) \, dx \) is said to be convergent if the corresponding [finite] limit exists and divergent if the [finite] limit does not exist.

• If \( f \) has a discontinuity at \( c \), where \( a < c < b \), and both \( \int_a^c f(x) \, dx \) and \( \int_c^b f(x) \, dx \) are convergent, then we define \( \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \).

Evaluate \( \int_0^3 \frac{dx}{x - 1} \):

• This is not an ordinary definite integral because there is a vertical asymptote \( x = 1 \).
• Improper integrals must be calculated in terms of limits!

\( \int_0^3 \frac{dx}{x - 1} = \int_0^1 \frac{dx}{x - 1} + \int_1^3 \frac{dx}{x - 1} \)

where \( \int_0^1 \frac{dx}{x - 1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x - 1} = \lim_{t \to 1^-} [\ln |x - 1|]_0^t = \lim_{t \to 1^-} (\ln|t - 1| - \ln|1|) = \lim_{t \to 1^-} \ln(1 - t) = -\infty \)

Since \( \int_0^1 \frac{dx}{x - 1} \) is divergent, so is \( \int_0^3 \frac{dx}{x - 1} \).

Comparison test for improper integrals:

• Suppose that \( f \) and \( g \) are continuous functions with \( f(x) \geq g(x) \geq 0 \) for \( x \geq a \).
• If \( \int_a^\infty f(x) \, dx \) is convergent, then \( \int_a^\infty g(x) \, dx \) is also convergent.
• If \( \int_a^\infty g(x) \, dx \) is divergent, then \( \int_a^\infty f(x) \, dx \) is also divergent.

Solutions:

\( \int_{-\infty}^0 \frac{1}{2x - 5} \, dx = \lim_{t \to -\infty} \int_t^0 \frac{1}{2x - 5} \, dx = \lim_{t \to -\infty} \left[ \frac{1}{2} \ln |2x - 5| \right]_t^9 = \lim_{t \to -\infty} \left[ \frac{1}{2} \ln 5 - \frac{1}{2} \ln |2t - 5| \right] = -\infty \)

divergent