EVALUATING DEFINITE INTEGRALS

Objective: Evaluate definite integrals using the evaluation theorem

If \( f \) is **continuous** on \([a, b]\), then the definite integral of \( f \) from \( x = a \) to \( x = b \) is

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = F(b) - F(a),
\]

where \( F \) is **any** antiderivative of \( f \).

Since \( F \) can be any antiderivative, we may choose to let \( C = 0 \)! If you choose \( C \) to be any nonzero real number, it will just cancel out in the evaluation process.

**Evaluation theorem:**

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

Example: Find \( \int_{-5}^{4} (x^2 - 4) \, dx \) and interpret the result in terms of areas.

\[
\int_{-5}^{4} (x^2 - 4) \, dx = \left[ \frac{x^3}{3} - 4x \right]_{-5}^{4} = \left[ \frac{4^3}{3} - 4(4) \right] - \left[ \frac{(-5)^3}{3} - 4(-5) \right] = 27
\]

1. \( \int_{-5}^{-2} (x^2 - 4) \, dx \approx M_6 = 26.9375 \)

\[
\int_{-5}^{-2} (x^2 - 4) \, dx = \left[ \frac{(-2)^3}{3} - 4(-2) \right] - \left[ \frac{(-5)^3}{3} - 4(-5) \right] = 27
\]

2. \( \int_{-2}^{2} (x^2 - 4) \, dx \approx M_8 = -10.75 \)

\[
\int_{-2}^{2} (x^2 - 4) \, dx = \left[ \frac{2^3}{3} - 4(2) \right] - \left[ \frac{-2^3}{3} - 4(-2) \right] = -\frac{32}{3}
\]

3. \( \int_{2}^{4} (x^2 - 4) \, dx \approx M_4 = 10.625 \)

\[
\int_{2}^{4} (x^2 - 4) \, dx = \left[ \frac{4^3}{3} - 4(4) \right] - \left[ \frac{2^3}{3} - 4(2) \right] = \frac{32}{3}
\]
4. \( \int_{-5}^{4} (x^2 - 4) \, dx \approx M_{18} = 26.8125 \)

\[
\int_{-5}^{4} (x^2 - 4) \, dx = 27 - \frac{32}{3} + \frac{32}{3} = 27
\]

*The definite integral equals the sum of the areas above the x-axis minus the areas below the x-axis*

Summary: To find the exact value of \( \int_{0}^{4} (x^2 - 4) \, dx \)

Apply equation 3 from Section 5.2 [p. 354], a lengthy tedious process

\[
\int_{0}^{4} (x^2 - 4) \, dx = \frac{16}{3}
\]

Use the evaluation theorem from Section 5.3 [p.366]

\[
\int_{0}^{4} (x^2 - 4) \, dx = \left[ \frac{x^3}{3} - 4x \right]_{0}^{4} = \left[ \frac{(4)^3}{3} - 4(4) \right] - \left[ \frac{(0)^3}{3} - 4(0) \right] = \frac{16}{3}
\]

Evaluate \( \int_{1}^{\sqrt{x}} dx \)

Evaluate \( \int_{\pi}^{2\pi} \cos \theta \, d\theta \)

The *indefinite* integral \( \int f(x) \, dx \) is a *function*.

The *indefinite* integral \( \int f(x) \, dx \) is the set of all *antiderivatives* of the function \( F(x) \), where \( f(x) = F(x) \).

Example: \( \int \frac{1}{x} \, dx = \ln|x| + C \)

C is called the *constant of integration*.
C may be any real number, but can only be evaluated if an *initial condition* is given.
The definite integral \( \int_{a}^{b} f(x) \, dx \) is a number.

Example: \( \int_{1}^{e} \frac{1}{x} \, dx = \left[ \ln|x| \right]_{1}^{e} = \ln(e) - \ln(1) = 1 \)

**KNOW** the table of indefinite integrals on page 369

These formulas are only valid on closed intervals where the function is **continuous**

\[ \int f(x) \, dx = F(x) \Leftrightarrow F'(x) = f(x) \]

**Total change theorem**

If \( y = F(x) \), then \( F'(x) \) represents the rate of change of \( F(x) \) with respect to \( x \), and \( F(b) - F(a) \) is the total change in \( y \) when \( x \) changes from \( a \) to \( b \)

\[ \int_{a}^{b} F'(x) \, dx = F(b) - F(a) \]

The (ever-popular) moving particle problem: A particle moves along a line with velocity of \( v(t) = (t^2 - t - 6) \, m/s \) at time \( t \).

Find the displacement of the particle from \( t = 1 \) s to \( t = 4 \) s

Displacement = change in location relative to starting point

\[ s(4) - s(1) = \int_{1}^{4} v(t) \, dt = \int_{1}^{4} (t^2 - t - 6) \, dt = \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_{1}^{4} = -\frac{9}{2} = -4.5 \, m \]

This means the particle is now located 4.5 m left of its starting point

Find the total distance traveled during this time period

Recall that speed = \(|v(t)| = |(t - 3)(t + 2)|\)

\( v(t) < 0 \) on \((1, 3)\) and \( v(t) > 0 \) on \((3, 4)\)

Total distance traveled = \( \int_{1}^{4} |v(t)| \, dt = -\int_{1}^{3} v(t) \, dt + \int_{3}^{4} v(t) \, dt = -\left( \int_{1}^{3} (t^2 - t - 6) \, dt + \int_{3}^{4} (t^2 - t - 6) \, dt \right) \)
\[
= - \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^3 + \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4 = \frac{61}{6} \approx 10.17 \text{ m}
\]

Particle has moved a total of about 10.17 m during the period from \( t = 1 \) s to \( t = 4 \) s.

Visualization of the motion of the particle

<table>
<thead>
<tr>
<th>time ( t ) in sec</th>
<th>3</th>
<th>2</th>
<th>4</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>displacement in m</td>
<td>-22/3</td>
<td>-31/6</td>
<td>-4.5</td>
<td>0</td>
</tr>
<tr>
<td>velocity in m/s</td>
<td>0</td>
<td>-4</td>
<td>6</td>
<td>-6</td>
</tr>
</tbody>
</table>

Find and interpret the displacement of the particle from \( t = 0 \) s to \( t = 5 \) s.

Find the total distance traveled during the first 5 s.

General motion of an object

If an object moves along a straight line with position function \( s(t) \) at time \( t \),

Then its velocity is \( v(t) = s'(t) \) at time \( t \), \( \text{(speed} = |v(t)|) \)

And \( \int_{t_1}^{t_2} v(t) \, dt = s(t_2) - s(t_1) \) is the **total change of position**, or **displacement**, of the particle during the period from time \( t_1 \) to time \( t_2 \).

If the velocity function of the object is \( v(t) \) at time \( t \),

Then its **acceleration** (change in velocity) is \( a(t) = v'(t) \) at time \( t \),

And \( \int_{t_1}^{t_2} a(t) \, dt = v(t_2) - v(t_1) \) is the **total change** in velocity from time \( t_1 \) to time \( t_2 \).
Some typical applications of the integral to find total change (see p. 371-2)

If \( V(t) \) = volume of water in a reservoir at time \( t \),

Then \( V'(t) \) is the rate at which water flows into (or out of) the reservoir at time \( t \),

And \( \int_{t_1}^{t_2} V'(t) \, dt = V(t_2) - V(t_1) \) is the total change in the amount of water in the reservoir between time \( t_1 \) and time \( t_2 \)

If \([C](t)\) is the concentration of the product C of a chemical reaction at time t

Then the rate of reaction is \( \frac{d}{dt} [C] \) at time \( t \),

And \( \int_{t_1}^{t_2} \frac{d[C]}{dt} = [C](t_2) - [C](t_1) \) is the total change in the concentration of C from time \( t_1 \) to time \( t_2 \).

If the mass of a rod measured from the left end to point \( x \) is \( m(x) \),

Then the linear density (change in mass per unit length) is \( \rho(x) = m'(x) \) at point \( x \),

And \( \int_a^b \rho(x) \, dx = m(b) - m(a) \) is the total mass of the segment of the rod that lies between \( x = a \) and \( x = b \).

The linear density of a rod of length 4 m is given by \( \rho(x) = 9 + 2\sqrt{x} \) measured in kilograms per meter, where \( x \) is measured in meters from the left end of the rod.

Find and interpret \( \rho(3) \).

Find the [exact] total mass of the rod.

Find the [exact] mass of the first 3 m of the rod, measured from the left end.

Find the [exact] mass of the rod between 1 m and 2 m, measured from left end.
If \( n(t) \) represents a population at time \( t \),

Then the **rate of growth** (or decay) of the population is \( \frac{dn}{dt} \) at time \( t \),

And \( \int_{t_1}^{t_2} \frac{dn}{dt} \) is the **total change** in the population during the period from time \( t_1 \) to time \( t_2 \).

If \( C(x) \) is the cost of producing \( x \) units of a commodity,

Then the **marginal cost** of producing the \((x+1)^{st}\) unit is \( C'(x) \),

And \( \int_{x_1}^{x_2} \frac{dC}{dt} = C(x_2) - C(x_1) \) is the total increase in cost when production is increased from \( x_1 \) units to \( x_2 \) units.

Power is the **rate of change** of energy: \( P(t) = E'(t) \).

The unit of measurement for \( \int_a^b f(x)\,dx \) is the product of the units for \( f(x) \) and the units for \( x \)

Example: A particle moves along a line with velocity of \( v(t) = (t^2 - t - 6) \) m/s at time \( t \).

Displacement = \( s(4) - s(1) = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt = -4.5 \) m,

because \( v(t) \) is in m/s and \( dt \) is in s, therefore the product is in \( \frac{m}{s} \) (s) = m.

\[ \int_1^4 \frac{1}{\sqrt{x}} \,dx = \int_1^4 x^{-1/2} \,dx = \left[ 2x^{1/2} \right]_1^4 = 2(\sqrt{4} - \sqrt{1}) = 2 \]

\[ \int_0^{2\pi} \cos \theta \,d\theta = \left[ \sin \theta \right]_0^{2\pi} = \sin 2\pi - \sin \pi = 0 - 0 = 0 \]

Displacement: \( s(5) - s(0) = \int_0^5 v(t)\,dt = \int_0^5 (t^2 - t - 6)\,dt = \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_0^5 = -\frac{m}{6} \)

After 5 s the particle is \( \frac{5}{6} \) m to the left of its position at \( t = 0 \) s.
Total distance traveled: \( \int_{0}^{5} |v(t)|dt = -\int_{0}^{3} v(t)dt + \int_{3}^{5} v(t)dt \)

\[ = -\int_{0}^{3} (t^2 - t - 6)dt + \int_{3}^{5} (t^2 - t - 6)dt \]

\[ = -\left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_{0}^{3} + \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_{3}^{5} = 13.5 + \frac{38}{3} = \frac{157}{6} \text{ m} \approx 26.17 \text{ m} \]

Particle has moved a total of about 26.17 m during the period from \( t = 0 \) s to \( t = 5 \) s.

Linear density at \( x = 3 \text{ m} = \rho(3) = 9 + 2\sqrt{3} \text{ kg/m} \). The mass of the rod is changing at a rate of \( 9 + 2\sqrt{3} \) kilograms per meter at the point which is 3 m from the left end of the rod.

Total mass \( = m(4) - m(0) = \int_{0}^{4} (9 + 2\sqrt{x})dx = \left[ 9x + \frac{4}{3}x^{3/2} \right]_{0}^{4} = \frac{140}{3} - 0 = \frac{140}{3} \text{ kg} \)

Mass of first 3 m from left end \( = m(3) - m(0) = \int_{0}^{3} (9 + 2\sqrt{x})dx = \left[ 9x + \frac{4}{3}x^{3/2} \right]_{0}^{3} = 27 + \frac{4}{3}(3^{3/2}) - 0 = 27 + 4\sqrt{3} \text{ kg} \)

Mass between 1m and 2 m \( = m(2) - m(1) = \int_{1}^{2} (9 + 2\sqrt{x})dx = \left[ 9x + \frac{4}{3}x^{3/2} \right]_{1}^{2} = 18 + \frac{4}{3}(2^{3/2}) - \left[ 9 + \frac{4}{3} \right] \approx 18 + \frac{8}{3}\sqrt{2} - \frac{31}{3} = \frac{23}{3} + \frac{8}{3}\sqrt{2} \text{ kg} \)