L'Hôpital's Rule

Recall way back in the semester - LIMITS!! They're back!
Why back and why now?
The reason is because now we can solve some limit problems that defeated us before - at least we had to resort to tables or a graph to find the limit. We could not eventually substitute, no matter what we did.
Now we have the derivative as a tool.

Here is one that gave us real trouble: \( \lim_{x \to 0} \frac{\sin x}{x} \)
If we tried to substitute \( x=0 \), we ended up with \( \frac{0}{0} \), which is an indeterminate form.

Now, however, with L'Hôpital's Rule, we can solve this limit.

The rule: If \( h(x) = \frac{f(x)}{g(x)} \) and \( \lim_{x \to c} \frac{f(x)}{g(x)} \) is an indeterminate form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \), then

\[
\lim_{x \to c} h(x) = \lim_{x \to c} \frac{f'(x)}{g'(x)}
\]

In English, the limit of the quotient of functions equals the limit of their derivatives (indeterminate conditions must be satisfied, substitution results in \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \)).

So, \( \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \frac{1}{1} = 1 \)

Here's another: \( \lim_{x \to \infty} \frac{3x^3 - 2x + 1}{4x^3 - 5x} = \lim_{x \to \infty} \frac{9x^2 - 2}{12x^2 - 5} \) (again) \( \frac{18x}{24x} = \lim_{x \to \infty} \frac{18}{24} = \frac{3}{4} \)

Other possibilities of an indeterminate form are \( 0 \cdot \infty, \frac{\infty}{0}, \frac{0}{0}, 1^- \)
Then, what?

The plan:

Form | Change to a quotient:
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* Product: \( 0 \cdot \infty \) | \( f(x)g(x) = \frac{f(x)}{1} \) or \( f(x)g(x) = \frac{g(x)}{1} \)
* Difference: \( \infty - \infty \) | \( f(x) - g(x) = \frac{h(x)}{\text{common denominator}} \)
* Powers: \( 0^0, \infty^0, 1^- \) | Let \( y = f(x)^{g(x)} \)
\[ \ln y = g(x) \ln f(x) \] Now it's a product, so work as above to find the limit \( L \)
Then, \( \ln y = L \)
So, \( y = e^L \)