Ex. The position of a particle moving on the $x$ axis is modeled by the equation $s(t) = 36t - 12t^2 + t^3$, $t \geq 0$, where $t$ is measured in seconds and $s(t)$ is measured in inches.

(a) What is the velocity of the particle at time $t$?
(b) What is the velocity after 2 seconds?
(c) When is the particle at rest?
(d) When is the particle moving forward (to the right)?
(e) Find the total distance traveled in the first 10 seconds.
(f) Draw a one-dimensional position diagram showing the path of the particle as it moves on the $x$-axis.
(g) Find the acceleration of the particle at time $t$ and after 2 seconds.
(h) Graph all three functions: position $s(t)$, velocity $s'(t)$ and acceleration $s''(t)$.
(i) When is the particle speeding up? When is it slowing down?

SOLUTION: Here’s what we’ll need for this problem …

Position function $s(t) = 36t - 12t^2 + t^3$, $t \geq 0$ (the inequality is just to let you know to only use positive time value)

velocity function $v(t) = s'(t) = 36 - 24t + 3t^2$ (first derivative)

acceleration function $a(t) = v'(t) = s''(t) = -24 + 6t$ (second derivative)

a) Velocity at time $t$ is $s'(t) = 36 - 24t + 3t^2$.

b) Velocity after 2 seconds is $v(2) = s'(2) = 36 - 24(2) + 3(2)^2 = 0$ inches / second

c) The particle is said to be “at rest” when the velocity is zero … so set velocity $= 0$ and solve for time $t$.  
$36 - 24t + 3t^2 = 0 \quad \Rightarrow \quad$ solutions are at $t = 2$ sec. and $t = 6$ sec.
This particle is at rest at $t = 2$ sec. and $t = 6$ sec.

d) “Right” and “up” should always be considered a positive vector velocity value. “Left” and “down” are negative vector velocity values.

This particle is moving right when $s'(t) > 0$ … look at the graph of $s'(t) = 36 - 24t + 3t^2$

The velocity is positive on these intervals: $(0, 2) \cup (6, \infty)$

This means the particle is moving right on the same intervals … $(0, 2) \cup (6, \infty)$ seconds.
e) Be careful with this one … in the first 10 seconds of its travel, the particle turns around twice … once at 2 seconds and again at 6 seconds. You need to find the total distance in intervals …

\[
\begin{align*}
  s(0) &= 0 \text{ inches} \quad \leftarrow \text{the particle starts at 0 inches.} \\
  s(2) &= 32 \text{ inches} \quad \leftarrow \text{the particle is 32 inches to the right after 2 seconds} \\
  s(6) &= 0 \text{ inches} \quad \leftarrow \text{the particle is back at 0 inches … where it started.} \\
  s(10) &= 160 \text{ inches} \quad \leftarrow \text{at 10 seconds, the particle is 160 inches to the right of its starting point.}
\end{align*}
\]

So the total distance traveled is 32 + 32 + 160 = 224 inches

f) The derivative graph helps to show the motion of the particle …

- moving right between \( t = 0 \) and \( t = 2 \) seconds …
- moving left between \( t = 2 \) and \( t = 6 \) seconds …
- moving right after 6 seconds

using the position values from part (e) we can represent the actually 1-dimensional motion of this particle …

\[
\begin{align*}
  t = 0 &\quad \rightarrow \quad t = 2 \\
  t = 6 &\quad \rightarrow \quad 32 \text{ inches}
\end{align*}
\]

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g) The acceleration after 2 seconds is \( a(2) = s''(2) = -24 + 6(2) = -12 \text{ inches/sec}^2 \)
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h)

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g) The particle is “speeding up” when its acceleration and velocity have the same sign …

- Speeding up on the time interval between \( t = 2 \) and \( t = 4 \) seconds … and again after \( t = 6 \) seconds to infinity.

The particle is slowing down when the velocity and acceleration have opposite signs

- Slowing down between \( t = 0 \) and \( t = 2 \) seconds … and between \( t = 4 \) and \( t = 6 \) seconds.
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Ex. A spherical balloon is being inflated. Find the rate of change of the volume, \( V = \frac{4\pi r^3}{3} \) with respect to the radius \( r \) when \( r \) is (a) 10 cm (b) 11 cm (c) 12 cm. What conclusions can you make?

SOLUTION:
The function \( V = \frac{4\pi r^3}{3} \) defines the volume of the balloon as a function of its radius.
The derivative of this function gives the rate of change of volume with respect to the radius.
\[
V = \frac{4\pi r^3}{3} \implies V' = \frac{4\pi}{3} (3r^2) = 4\pi r^2
\]
(a) \( V'(10) = 4\pi (10)^2 = 400\pi \text{ cm}^3 \text{ per cm} \)
(\text{meaning the volume is growing at a rate of } 400\pi \text{ cm}^3 \text{ per cm of radius length when the radius is 10 cm long.})
(b) \( V'(11) = 4\pi (11)^2 = 484\pi \text{ cm}^3 \text{ per cm} \)
(c) \( V'(12) = 4\pi (12)^2 = 576\pi \text{ cm}^3 \text{ per cm} \)
Since the rate of change is increasing as the radius increases … the volume is obviously growing, but it is growing at an increasing rate.

Ex. Boyle’s Law states that when a sample of gas is compressed at a constant temperature, the product of the pressure (in atmospheres, atm) and the volume remains constant: \( PV = C \).
(a) Find the rate of change of the pressure with respect to the volume.
(b) What does the sign of the result from part (a) signify?

SOLUTION:
(a) \( PV = C \implies \text{ rewrite so that pressure is a function of volume } \implies P = \frac{C}{V} \)
The rate of change of pressure with respect to volume is \( P' = C(-V^{-2}) \implies P' = -\frac{C}{V^2} \)
(b) The negative sign indicates the fact that pressure (as a function of volume) is a decreasing function.
The more volume increases, the more pressure decreases which is also suggested by the inverse proportional relation \( P = \frac{C}{V} \).
Ex. Assume a certain physical phenomenon is modeled by the equation $F = \frac{5A^2}{\sqrt{BC}}$. Find …

(a) The rate of change of $F$ with respect to $A$, $\frac{dF}{dA}$ (this means assume that $B$ and $C$ are constants)

(b) The rate of change of $F$ with respect to $B$, $\frac{dF}{dB}$ (this means assume that $A$ and $C$ are constants)

SOLUTION:

(a) Isolate the variable $A$ \[ F = \frac{5}{\sqrt{BC}} \cdot A^2 \Rightarrow \frac{dF}{dA} = \frac{5}{\sqrt{BC}} \cdot (2A) = \frac{10A}{\sqrt{BC}} \]

(b) Isolate the variable $B$ \[ F = \frac{5A^2}{\sqrt{C}} \cdot B^{-1/2} \Rightarrow \frac{dF}{dB} = \frac{5A^2}{\sqrt{C}} \cdot \left(\frac{1}{2} B^{-3/2}\right) = \frac{5A^2}{\sqrt{C}} \cdot \frac{1}{2\sqrt{B^3}} = \frac{5A^2}{2\sqrt{B^3C}} \]