Intro to Limits

What is a Limit?

Limits are tools you will be using extensively throughout calculus (1 and 2) for the purpose of evaluating various functions and expressions at inputs which may or may not be in the function’s or expression’s domain.

Definition: Suppose the function \( f(x) \) is defined for all \( x \) values near \( x = a \) except possibly at \( x = a \) itself (i.e. \( f(x) \) has \( y \) values of output for all \( x \) values of input near but not necessarily at \( x = a \)). Then the statement

\[
\lim_{x \to a} f(x) = L \quad \text{(read 'The limit of } f(x) \text{ as } x \text{ approaches } a \text{ equals } L')
\]

means that the \( y \)-values (our output values) of \( f(x) \) get very close to the \( y \)-value \( L \) the closer the \( x \) input values get to \( a \).

Ex) Use a table to evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \).

Is the function defined at \( x = 0 \)? \textbf{No}

Inputs approaching \( x = 0 \) from the right

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \frac{\sin x}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.9983417</td>
</tr>
<tr>
<td>0.01</td>
<td>0.99998333</td>
</tr>
<tr>
<td>0.001</td>
<td>0.99999983</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.9999999983</td>
</tr>
</tbody>
</table>

This is referred to as a \textbf{one-sided limit}. We’re approaching 0 from the right. Specifically it’s denoted as

\[
\lim_{x \to 0^+} \frac{\sin x}{x} = 1
\]

Inputs approaching \( x = 0 \) from the left

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \frac{\sin x}{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>0.9983417</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.99998333</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.99999983</td>
</tr>
<tr>
<td>-0.0001</td>
<td>0.9999999983</td>
</tr>
</tbody>
</table>

Here we’re approaching from the left. This one-sided limit is denoted as ...

\[
\lim_{x \to 0^-} \frac{\sin x}{x} = 1
\]

\[
\lim_{x \to 0^+} \frac{\sin x}{x} = 1
\]

\[
\lim_{x \to 0^-} \frac{\sin x}{x} = 1
\]

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]
A (two-sided) limit \( \lim_{x \to a} f(x) \) is said to exist when each of the one-sided limits are equal: 

\[
\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)
\]

Therefore, \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \)

Look at the graph of \( f(x) = \frac{\sin x}{x} \).

Since the limit focuses on \( x = 0 \), zoom in close to that point on the graph.

Ex) Evaluate \( \lim_{x \to 16} \frac{\sqrt{x} - 4}{16 - x} \) using an appropriately chosen table of values.

<table>
<thead>
<tr>
<th>X</th>
<th>( f(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.9</td>
<td>-.12519592525</td>
</tr>
<tr>
<td>15.99</td>
<td>-.12501953736</td>
</tr>
<tr>
<td>15.999</td>
<td>-.1250019532</td>
</tr>
<tr>
<td>16.1</td>
<td>-.12480529548</td>
</tr>
<tr>
<td>16.01</td>
<td>-.12498647985</td>
</tr>
<tr>
<td>16.001</td>
<td>-.1249980469</td>
</tr>
</tbody>
</table>

\[
\lim_{x \to 16^-} \frac{\sqrt{x} - 4}{16 - x} = -0.125 = -\frac{1}{8}
\]

\[
\lim_{x \to 16^+} \frac{\sqrt{x} - 4}{16 - x} = -0.125 = -\frac{1}{8}
\]

Therefore, \( \lim_{x \to 16} \frac{\sqrt{x} - 4}{16 - x} = -\frac{1}{8} \)

\[
\lim_{x \to 16} \frac{\sqrt{x} - 4}{16 - x} = \lim_{x \to 16} \frac{\sqrt{x} - 4}{(4 - \sqrt{x})(4 + \sqrt{x})} = \lim_{x \to 16} \frac{-1(4 - \sqrt{x})}{(4 - \sqrt{x})(4 + \sqrt{x})} = \lim_{x \to 16} \frac{-1}{4 + \sqrt{x}} = \frac{-1}{4 + 4} = \frac{-1}{8}
\]

BE CAREFUL NOT TO ASSUME SMALL OUTPUTS ARE 0!
What do these tables allow you to conclude about \( \lim_{x \to -3} \frac{x^2 - 2x - 15}{|x + 3|} \) ? Does Not Exist

Sketch the graph of \( f(x) = \frac{x^2 - 2x - 15}{|x + 3|} \)
Limits can also be evaluated using a function’s graph.

As bizarre as this function appears, it is still a function. It passes the vertical line test.

Ex) For the function $f(x)$ graphed at the right, evaluate the following.
The axes are scaled so each tic mark represents a unit of 1.

REMEMBER!
When evaluating a limit, you only reply with what the $y$ outputs are approaching.
When evaluating a function output value, you look for a $y$ coordinate on the graph.

a) $\lim_{x \to -1} f(x) = -4$

b) $\lim_{x \to 0} f(x) = 2$

c) $\lim_{x \to 1} f(x) = \text{Does Not Exist (Undefined)}$

d) $\lim_{x \to 4} f(x) = -1.4$

e) $f(4) = \text{undefined}$

f) $\lim_{x \to 7} f(x) = -1$

g) $f(7) = 4$

h) $\lim_{x \to 9} f(x) = -2.4$

i) $f(9) = 6$