Squeezing Theorem and Trigonometric Limits

Before we delve too deeply into trigonometric limits, I need to show you a special limit theorem which offers up a handy trick.

The Squeezing Theorem
The function $f(x)$ is bounded by the functions $L(x)$ and $U(x)$ so that they satisfy the inequality $L(x) \leq f(x) \leq U(x)$ in some region near the value $x = a$.

If $\lim_{x \to a} L(x) = C$ and $\lim_{x \to a} U(x) = C$, then since $f(x)$ is ‘squeezed’ between the functions $L(x)$ and $U(x)$, we can conclude that $\lim_{x \to a} f(x) = C$ also.

In my experience, the Squeezing Theorem is most often used in limits involving trigonometric functions.

Ex) Use Squeezing Theorem to evaluate $\lim_{x \to 0} x^2 \sin(\frac{10}{x})$.

Properly using the Squeezing Theorem is almost like setting up a **mathematical proof**

Begin with the sine expression:
Ex) Use the squeeze theorem to evaluate \( \lim_{x \to 0^+} \sqrt{x} e^{\cos\left(\frac{\pi}{x}\right)} \)

Ex) Use the squeeze theorem to evaluate \( \lim_{t \to 0^+} \sin t \tan^{-1} (\ln t) \)
Special Trigonometric Limits

There are 2 special limits involving sine and cosine which we'll use again when we study derivatives. They are ...

\[ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0 \]

YOU WILL WANT TO COMMIT THESE TO MEMORY!!!

Ex) Evaluate \( \lim_{t \to 0} \frac{\sec t \sin t}{t} \).

Ex) Evaluate \( \lim_{x \to 0} \frac{\sin x}{4x} \).

Ex) Evaluate \( \lim_{x \to 0} \frac{\sin 8x}{3x} \).

HINT: The argument of sine and the denominator must be the EXACT same expression in order to properly use the special trig limit!
Ex) Evaluate the limit \( \lim_{x \to 0} \frac{\cos x - \cos^2 x}{x} \)