The Derivative of Sine

To differentiate the function $f(x) = \sin x$, we'll start with its difference quotient limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

The expression $\sin(x+h)$ can be expanded using a formula from trig:

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

Rearrange the terms up top and then separate the numerator:

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

After some further algebraic rearranging, this limit becomes

$$\lim_{h \to 0} \sin(x)\cdot \left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\cdot \left(\frac{\sin(h)}{h}\right)$$

Since $\lim_{h \to 0} \left(\frac{\cos(h) - 1}{h}\right) = 0$ and $\lim_{h \to 0} \left(\frac{\sin(h)}{h}\right) = 1$, the difference quotient limit becomes

$$\lim_{h \to 0} \sin(x)\cdot \left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\cdot \left(\frac{\sin(h)}{h}\right) = \sin(x)\cdot (0) + \cos(x)\cdot (1) = \cos x$$

After all the calculations, we have finally shown that the derivative of $f(x) = \sin x$ is $f'(x) = \cos x$ (Now, let's work the derivative of $f(x) = \cos x$)

### Derivatives of the Trigonometric Functions (KNOW THEM WELL!)

<table>
<thead>
<tr>
<th>$\frac{d}{dx} (\sin x) = \cos x$</th>
<th>$\frac{d}{dx} (\tan x) = \sec^2 x$</th>
<th>$\frac{d}{dx} (\sec x) = \sec x \tan x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d}{dx} (\cos x) = -\sin x$</td>
<td>$\frac{d}{dx} (\cot x) = -\csc^2 x$</td>
<td>$\frac{d}{dx} (\csc x) = -\csc x \cot x$</td>
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</tbody>
</table>
Prove all the remaining Trig function derivatives!

1. Differentiate \( f(x) = 5x^3 \sin x \)

2. Differentiate \( g(x) = \frac{\sec x}{1 + \tan x} \)

3. For what values of \( x \) does \( f(x) = x - 2\cos x \) have a horizontal tangent?

4. A mass on a spring vibrates horizontally on a smooth level surface as shown in the figure at the left. The equation of motion for the mass is \( x(t) = 8\sin t \), where \( x \) is in centimeters and \( t \) is in seconds.
   a) Find the velocity and acceleration at time \( t \).
   b) Find the position, velocity and acceleration of the mass at \( t = 2\pi / 3 \) seconds.
      What is its direction of motion at that time?
      Is it speeding up or slowing down?

5. What is the 51st derivative of \( y = \sin x \)?