Chapter 3
Mathematics of Finance

Section 3
Future Value of an Annuity; Sinking Funds

Definition of Annuity

- An **annuity** is any sequence of equal periodic payments.
- An **ordinary annuity** is one in which payments are made at the end of each time interval. If for example, $100 is deposited into an account every quarter (3 months) at an interest rate of 8% per year, the following sequence illustrates the growth of money in the account after one year:

\[
100 + 100 \left(1 + \frac{0.08}{4}\right) + 100(1.02)(1.02) + 100(1.02)(1.02)(1.02)
\]

This amount was just put in at the end of the 4th quarter, so it has earned no interest.

General Formula for Future Value of an Annuity

\[
FV = PMT \left(\frac{(1 + i)^n - 1}{i}\right)
\]

where
- \(FV\) = future value (amount)
- \(PMT\) = periodic payment
- \(i\) = rate per period
- \(n\) = number of payments (periods)

Example

- Suppose a $1000 payment is made at the end of each quarter and the money in the account is compounded quarterly at 6.5% interest for 15 years. How much is in the account after the 15 year period?
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- **Solution:**

  \[ FV = PMT \left( \frac{(1+i)^n - 1}{i} \right) \]

  \[ FV = 1000 \left( \frac{1 + \frac{0.065}{4}}{4} \right)^{4(15)} - 1 \]

  \[ FV = 1000 \left( \frac{1 + \frac{0.065}{4}}{4} \right)^{60} - 1 \]

  \[ FV = 1000 \cdot 1.065^{60} - 1 \]

  \[ FV = 1000 \cdot 2.30083 - 1 \]

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  \[ FV = 2300.83 - 1 \]

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Amount of Interest Earned

- How much interest was earned over the 15 year period?

**Solution:**

Each periodic payment was $1000. Over 15 years, 15(4) = 60 payments were made for a total of $60,000. Total amount in account after 15 years is $100,336.68. Therefore, amount of accrued interest is $100,336.68 - $60,000 = $40,336.68.
**Balance in the Account at the End of Each Period**

This graph displays the balance of the account at the end of each quarter.

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**Sinking Fund**

- **Definition:** Any account that is established for accumulating funds to meet future obligations or debts is called a *sinking fund*.
- The *sinking fund payment* is defined to be the amount that must be deposited into an account periodically to have a given future amount.

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**Sinking Fund Payment Formula**

To derive the sinking fund payment formula, we use algebraic techniques to rewrite the formula for the future value of an annuity and solve for the variable PMT:

\[
FV = PMT \left( \frac{(1+i)^n - 1}{i} \right)
\]

\[
FV \left( \frac{i}{(1+i)^n - 1} \right) = PMT
\]

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**Sinking Fund Sample Problem**

How much must Harry save each month in order to buy a new car for $12,000 in three years if the interest rate is 6% compounded monthly?
**Sinking Fund**

**Sample Problem Solution**

How much must Harry save each month in order to buy a new car for $12,000 in three years if the interest rate is 6% compounded monthly?

**Solution:**

\[
FV \left( \frac{i}{(1+i)^n - 1} \right) = PMT
\]

\[
12000 \left[ \frac{0.06}{12} \right] = pmt = 305.06
\]

**Approximating Interest Rates**

**Example**

Mr. Ray has deposited $150 per month into an ordinary annuity. After 14 years, the annuity is worth $85,000. What annual rate compounded monthly has this annuity earned during the 14 year period?

**Solution:**

Use the \( FV \) formula: Here \( FV = 85,000, PMT = $150 \) and \( n \), the number of payments is \( 14(12) = 168 \).

Substitute these values into the formula. Solution is approximated graphically.

\[
FV = PMT \left( \frac{(1+i)^n - 1}{i} \right)
\]

\[
a = 85,000, \quad 150 = \frac{(1+i)^{168} - 1}{i}
\]

\[
y = \frac{(1+x)^{168} - 1}{x} = \frac{85,000}{150} = 566.67
\]

Graph each side of the last equation separately on a graphing calculator and find the point of intersection.
The monthly interest rate is about 0.01253 or 1.253%.
The annual interest rate is about 15%.