4.9 Antiderivatives

Objective: Find a function whose derivative is a known function.

I. A function $F$ is called an antiderivative of $f$ on an interval $I$ if for all $x$ in $I$,

$$F'(x) = f(x).$$

A. Suppose $f(x) = 3x^2$

1. $F(x) = x^3$ is an antiderivative of $f(x)$ because $F'(x) = 3x^2 = f(x)$.
2. $F(x) = x^3 + \pi$ is an antiderivative of $f(x)$ because $F'(x) = 3x^2 = f(x)$.
3. $F(x) = x^3 - 4e^x$ is an antiderivative of $f(x)$ because $F'(x) = 3x^2 = f(x)$.
4. $F(x) = x^3 + C$ is the most general antiderivative of $f(x)$, where $C$ is an arbitrary constant

B. Suppose $f(x) = x^n$

1. $F(x) = \frac{1}{5} x^5 + C$ is the most general antiderivative of $F(x) = x^4$ because

$$F'(x) = 5 \left( \frac{1}{5} x^{5-1} \right) = x^4 = f(x).$$

2. $F(x) = \frac{1}{n+1} x^{n+1} + C$ is the most general antiderivative of $f(x) = x^n$

because $F'(x) = (n + 1) \left( \frac{1}{n+1} x^{(n+1)-1} \right) = x^n = f(x)$.

C. Antidifferentiation formulas

<table>
<thead>
<tr>
<th>Function</th>
<th>A particular antiderivative</th>
<th>The general antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$cf(x)$</td>
<td>$cF(x) + 5$</td>
<td>$cF(x) + C$</td>
</tr>
<tr>
<td>$f(x) + g(x)$</td>
<td>$F(x) + G(x) - 7e$</td>
<td>$F(x) + G(x) + C$</td>
</tr>
<tr>
<td>$x^n, n \neq 1$</td>
<td>$\frac{x^{n+1}}{n+1}$</td>
<td>$\frac{x^{n+1}}{n+1} + C$</td>
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<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
<td>x</td>
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<tr>
<td>(e^x)</td>
<td>(e^x - \ln 4)</td>
<td>(e^x + C)</td>
</tr>
<tr>
<td>(\cos x)</td>
<td>(\sin x)</td>
<td>(\sin x + C)</td>
</tr>
<tr>
<td>(\sin x)</td>
<td>(- \cos x)</td>
<td>(- \cos x + C)</td>
</tr>
<tr>
<td>(\sec^2 x)</td>
<td>(\tan x)</td>
<td>(\tan x + C)</td>
</tr>
<tr>
<td>(\frac{1}{\sqrt{1 - x^2}})</td>
<td>(\sin^{-1} x)</td>
<td>(\sin^{-1} x + C)</td>
</tr>
<tr>
<td>(\frac{1}{1 + x^2})</td>
<td>(\tan^{-1} x)</td>
<td>(\tan^{-1} x + C)</td>
</tr>
</tbody>
</table>

D. Find the [general] antiderivative of

1. \(f(x) = 2 + \sqrt{2}\)   
   \(F(x) = 2x + \frac{5}{8}x^{\frac{3}{2}} + C\)

2. \(f(x) = 5\sec^2 x + 7x^2 + 4x^{\frac{3}{2}} - \frac{e}{x} + 1\)   
   \(F(x) = 5\tan x + \frac{1}{3}7x^3 + \frac{5}{8}4x^{\frac{5}{2}} - e\ln|x| + x + C\)

3. Remember to check your work by taking the derivative of your answer; this should get you back to the original problem!

II. An equation that involves the derivative(s) of a function is called a **differential equation**.

A. Find \(f(x)\) if \(f'(x) = e^x + \frac{20}{1 + x^2}\) and \(f(0) = -2\)

1. \(f(x) = e^x + 20\tan^{-1} x + C\) is the **general solution**.

2. \(f(0) = e^0 + 20\tan^{-1} 0 + C = -2 \rightarrow C = -2 - 1 - 0 = -3\)

3. \(f(x) = e^x + 20\tan^{-1} x - 3\) is the **particular solution**.

B. Find \(f(x)\) if \(f''(x) = 12x^2 + 6x - 4\), if \(f(0) = 4\) and \(f(1) = 1\)

1. \(f'(x) = 12 \cdot \frac{x^2}{3} + 6 \cdot \frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C\)
2. \[ f(x) = 4 \cdot \frac{x^4}{4} + 3 \cdot \frac{x^3}{3} - 4 \cdot \frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D \]
   a. \( f(0) = 0 + 0 - 0 + 0 + D = 4 \rightarrow D = 4 \)
   b. \( f(1) = 1 + 1 - 2 + C + 4 = 1 \rightarrow C = -3 \)
   c. \( f(x) = x^4 + x^3 - 2x^2 + 4x - 3 \)

III. Sketch the graph of \( F(x) \) if \( f(x) = \sqrt{1 + x^3} - x \) and \( F(-1) = 0 \)
   A. We do not know how to find an antiderivative of \( f(x) \).
   B. We could draw the graph of \( f \) and use it to approximate the graph of \( F(x) \)
      [see Example 4 in Sec 2.10]
   C. See page 336 for the direction field for \( f(x) \)
      1. Since \( f(0) = 1 \), the graph of \( F(x) \) has slope 1 when \( x = 0 \); draw a short tangent
         segment with slope 1, centered at \( x = 0 \).
      2. Since \( f(1) = .4 \), the graph of \( F(x) \) has slope .4 when \( x = 1 \); draw a short tangent
         segment with slope .4, centered at \( x = 1 \).
      3. Repeat this process for several other values of \( x \).
      4. Each segment in the direction field indicates the direction in which the curve
         \( y = F(x) \) proceeds at that point.
   D. The initial condition \( F(-1) = 0 \) means that we should start at the point \((-1, 0)\) and draw
      the graph of \( F(x) \) so that it follows the directions of the tangent segments.

IV. Rectilinear motion
   A. If \( s = f(t) \) is the position function of a particle, then the velocity function is \( v(t) = s'(t) \);
      the position function is an antiderivative of the velocity function!
   B. If \( v = v(t) \) is the velocity function, then the acceleration function is \( a(t) = v'(t) \);
      the velocity function is an antiderivative of the acceleration function.

V. Example 7 on p. 337
   A ball is thrown upward with a speed of 48 ft/s from a height of 432 ft.
   A. Find its height above the ground \( t \) seconds later.
      1. Motion is vertical; choose upward to be the positive direction
      2. The only acceleration is the force of gravity, which is pulling downward
         1. \( a(t) = \frac{dv}{dt} = -32 \text{ ft}/\text{s}^2 \)
         2. \( v(t) = -32t + C \) and \( v_o = 48 \rightarrow C = 48 \)
         3. \( v(t) = -32t + 48 \)
B. When does it reach its maximum height?
   1. Maximum height → $v(t) = 0$
   2. $-32t + 48 = 0$ → $t = 1.5$ s

C. When does it hit the ground?
   1. $v(t) = \frac{ds}{dt} = -32t + 48$
   2. $s(t) = -16t^2 + 48t + D$ and $s(0) = 432$ → $D = 432$
   3. $s(t) = -16t^2 + 48t + 432$
   4. When the ball hits the ground $s(t) = 0$, so $-16t^2 + 48t + 432 = 0$, so
      $$t = \frac{3 \pm 3\sqrt{13}}{2}$$
   5. The ball hits the ground after approximately 6.9 seconds.