APPLICATIONS OF INTEGRATION

6.1 More About Areas

Objective: Find the area between two curves

I. The area between the graph of a continuous curve on \([a, b]\) and the x-axis

A. If \(f(x) \geq 0\) on \([a, b]\), then \(A = \int_a^b f(x)dx\)

B. If \(f(x) \leq 0\) on \([a, b]\), then \(A = \int_a^b f(x)dx = -\int_a^b f(x)dx = \int_b^a f(x)dx\)

II. Find the shaded area:

\[
A = \int_a^b f(x)dx + \int_b^c f(x)dx
\]

III. Find the area between the graphs of \(y = \sin(x)\) and \(y = \cos(x)\) on \(\left[\frac{\pi}{2}, \pi\right]\)

\[
A = \int_{\pi/2}^\pi (\sin x)dx - \int_{\pi/2}^\pi (\cos x)dx = \left[\frac{\sin x}{x}\right]_{\pi/2}^\pi - \left[\frac{\cos x}{x}\right]_{\pi/2}^\pi = [\cos x - \sin x]_{\pi/2}^\pi = \cos(\pi) - \cos(\pi/2) + \sin(\pi/2) - \sin(\pi) = 2
\]

IV. Find the area between the graphs of \(y = \sin(x)\) and \(y = \cos(x)\) on \([0, \pi]\)

A. The area between the graph of \(y = \sin(x)\) and the x-axis is \(A_1 = \int_0^\pi (\sin x)dx\)

B. The area between the graph of \(y = \cos(x)\) and the x-axis is
\[ A_2 = \int_{0}^{\pi/2} \cos(x)dx - \int_{\pi/2}^{\pi} \cos(x)dx \]

C. The only point where \( \sin(x) = \cos(x) \) on \([0, \pi]\) is at \( x = \frac{\pi}{4} \)

D. The shaded area is
\[
\begin{align*}
A &= \int_{0}^{\pi/4} \cos(x)dx - \int_{0}^{\pi/4} \sin(x)dx + \int_{\pi/4}^{\pi} \sin(x)dx - \int_{\pi/4}^{\pi/2} \cos(x)dx \\
&= \frac{\sqrt{2}}{2} - \left(1 - \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2} + 1\right) - \left(1 - \frac{\sqrt{2}}{2}\right) - (-1) = 2\sqrt{2} \\
&= \int_{0}^{\pi/4} [\cos(x) - \sin(x)]dx + \int_{\pi/4}^{\pi} [\sin(x) - \cos(x)]dx = (\sqrt{2} - 1) + (\sqrt{2} + 1) = 2\sqrt{2}
\end{align*}
\]

E. If \( f(x) \) and \( g(x) \) are continuous and \( f(x) \geq g(x) \) on \([a, b]\), then the area bounded by the graphs of \( f(x), g(x) \), \( x = a \), and \( x = b \) is \( \int_{a}^{b} [f(x) - g(x)]dx \) “top” – “bottom”

V. To find the area between the graphs of two continuous functions \( f(x) \) and \( g(x) \) on \([a, b]\), where \( f(x) \geq g(x) \), draw a typical rectangle, and label its width and height.

A. \( y = \cos(x) \)

B. The area between \( y = \cos(x) \) and \( y = \sin(x) \) on \([\pi/2, \pi]\) is \( \int_{\pi/2}^{\pi} [\sin(x) - \cos(x)]dx = 2 \)

VI. Find the area between the graphs of \( x = y^2 \) and \( y - x = 2 \) on \(-2 \leq y \leq 3\)

A. \( f(y) = y^2 \)

B. The area between \( x = y^2 \) and \( x = y - 2 \) is
\[
\int_{-2}^{3} [y^2 - (y - 2)]dy = \int_{-2}^{3} (y^2 - y + 2)dy = \frac{115}{6}
\]

C. If \( f(y) \) and \( g(y) \) are continuous and \( f(y) \geq g(y) \) on \([c, d]\), then the area bounded by the graphs of \( f(y), g(y) \), \( y = c \), and \( y = d \) is \( \int_{c}^{d} [f(y) - g(y)]dy \) “right” – “left”