APPLICATIONS OF INTEGRATION

6.2 Volumes

Objective: Find volume of solids of revolution

I. Definition of volume

A. S is a solid which lies between \( x = a \) and \( x = b \)
B. Cross-sectional area of S in plane \( P_x \), through \( x \) and perpendicular to \( x \)-axis is \( A(x) \)
C. A is a continuous function
D. \( V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta x = \int_{a}^{b} A(x) \, dx \)

II. A solid of revolution is formed by revolving a region in a plane about a line in the plane, called the axis of revolution.

III. Disk method: Find the volume of the solid of revolution generated by revolving the region formed by \( y = \frac{1}{2}x^3 - 2x^2 + x + 4 \), \( x = 0 \), \( x = 3 \), and the \( x \)-axis about the \( x \)-axis.

A. Consider a regular partition on \([0, 3]\) with \( \Delta x = \frac{1}{4} \)
B. Rotate area about \( x \)-axis
C. Each rectangle generates a circular disk
D. Volume of a disk = \( \pi r^2 h \)
E. Radius of a disk = \( f(x) \)
F. Height (thickness) of a disk = \( \Delta x \)
G. Volume of a disk = \( \pi (\text{radius})^2 (\text{thickness}) \)
H. Volume of a disk = \( \pi \left( \frac{1}{2}x^3 - 2x^2 + x + 4 \right)^2 \Delta x \)

I. \( V_L = \pi \left[ f(0)^2 + f\left(\frac{1}{4}\right)^2 + f\left(\frac{2}{4}\right)^2 + f\left(\frac{3}{4}\right)^2 + f\left(\frac{4}{4}\right)^2 + f\left(\frac{5}{4}\right)^2 + f\left(\frac{6}{4}\right)^2 \right. \\
\left. + f\left(\frac{7}{4}\right)^2 + f\left(\frac{8}{4}\right)^2 + f\left(\frac{9}{4}\right)^2 + f\left(\frac{10}{4}\right)^2 + f\left(\frac{11}{4}\right)^2 \right] \left(\frac{1}{4}\right) \)

\( V_L \approx 28.35\pi, \quad V_R \approx 25.91\pi, \quad V_M \approx 27.10\pi \)
\[ J. \quad V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta x = \int_{a}^{b} A(x) \, dx = \int_{a}^{b} \pi [f(x)]^2 \, dx = \int_{0}^{3} \pi \left( \frac{1}{2} x^3 - 2x^2 + x + 4 \right)^2 \, dx \]
\[
= \int_{0}^{3} \pi \left( \frac{x^6}{4} - 2x^5 + 5x^4 - 15x^2 + 8x + 16 \right) \, dx \\
= \pi \left[ \frac{x^7}{28} - \frac{x^6}{3} + x^5 - 5x^3 + 4x^2 + 16x \right]_{0}^{3} = \frac{759}{28} \pi 
\]

IV.  Find the volume of the solid of revolution formed by rotating the region bounded by \( y = x \), \( y = 3 \), and \( x = 0 \) about the \( y \)-axis.

A.  Thickness = \( \Delta y \)
B.  Radius = \( f(y) = y \)
C.  Volume of a disk = \( \pi y^2 \Delta y \)
D.  \( V = \int_{0}^{3} \pi y^2 \, dy = \pi \left[ \frac{y^3}{3} \right]_{0}^{3} = 9\pi \)

V.  Washer method: Find volume of solid obtained by rotating the region enclosed by the curves \( y = x \) and \( y = x^2 \) about the line \( y = 2 \)

A.  Thickness = \( \Delta x \)
B.  Cross-section is an annulus [ring or washer]
C.  Outer radius = \( 2 - x^2 \); inner radius = \( 2 - x \)
D.  Volume of a washer = \( [\pi (2 - x^2)^2 - \pi (2 - x)^2] \Delta x \)
E.  \( V = \int_{0}^{1} \pi [(2 - x^2)^2 - (2 - x)^2] \, dx = \pi \int_{0}^{1} (x^4 - 5x^2 + 4x) \, dx = \frac{8\pi}{15} \)

VI.  Summation of cross-sections may also be used to find volumes of solids which are not revolved about an axis: see Example 6 on p. 460