1. A square metal plate is placed in a furnace. The quick temperature change causes the metal plate to expand so that its surface area increases and its thickness decreases. If the side length of the plate is increasing at 2 mm per second, find the rate the area of the plate is increasing when the side length is 100 mm.

   a. Assign symbols to all quantities that are functions of time. Let \( t \) be the time in seconds.
      Let \( s \) be the side length of the square plate in mm.
      Let \( A \) be the area of the plate in mm\(^2\).

   b. Write the given information & the rate you are trying to find as appropriate derivatives.
      Given:
      \[ \frac{ds}{dt} = 2 \text{ mm/s} \]
       
      Find:
      \[ \frac{dA}{dt} \text{ when } s = 100 \text{ mm} \]

   c. Write an equation that relates the various quantities of the problem.
      \[ A = s^2 \]

   d. Implicitly differentiate both sides of equation with respect to time, \( t \).
      \[ \frac{dA}{dt} = 2s \frac{ds}{dt} \]

   e. Substitute given information into resulting equation and solve for the unknown rate.
      \[ \frac{dA}{dt} \bigg|_{s=100} = 2(100)(2) = 400 \text{ mm}^2/\text{s} \]

2. A pebble is thrown into a pond causing concentric circular ripples. The outermost ripple is increasing at a constant rate of 2 feet per second. When the radius is 4 feet, at what rate is the total area of the disturbed water changing?

   a. Assign symbols to all quantities that are functions of time. Let \( t \) be the time in seconds.
      Let \( r \) be the radius of the ripple and \( A \) be the enclosed area.

   b. Write the given information & the rate you are trying to find as appropriate derivatives.
      Given:
      \[ \frac{dr}{dt} = 2 \text{ ft/s} \]
      
      Find:
      \[ \frac{dA}{dt} \text{ when } r = 4 \text{ ft} \]

   c. Write an equation that relates the various quantities of the problem.
      \[ A = \pi r^2 \]

   d. Implicitly differentiate both sides of equation with respect to time, \( t \).
      \[ \frac{dA}{dt} = \pi \left(2r \frac{dr}{dt}\right) = 2\pi r \frac{dr}{dt} \]

   e. Substitute given information into resulting equation and solve for the unknown rate.
      \[ \frac{dA}{dt} \bigg|_{r=4} = 2\pi(4)(2) = 16\pi \text{ ft}^2/\text{s} \]

3. At a sand and gravel plant, sand is falling off a conveyor and onto a conical (cone-shaped) pile at a rate
of 10 cubic feet per minute. The diameter of the base of the cone is approximately 3 times the altitude (height). At what rate is the height of the pile changing when the pile is 15 feet high?

a. Draw a diagram & Assign symbols to all quantities (there are THREE in this case: The height, base radius, & volume of the conical pile) that are functions of time. Let \( t \) be the time in minutes.

Let \( h \) be the height in feet and \( r \) be the base radius in feet (see diagram below). Let \( V \) be the volume of the sand in cubic feet.

b. Write the given information & the rate you are trying to find as appropriate derivatives.

Given:

\[
\frac{dV}{dt} = 10 \text{ ft}^3/\text{min}
\]

Find:

\[
\frac{dh}{dt} \text{ when } h = 15 \text{ ft}
\]

c. Write an equation that relates the various quantities of the problem. Use information provided to eliminate one of the variables by substitution so that equation has only two variables (HINT: only volume & height should be left).

Equation that relates \( V \), \( h \), and \( r \):

\[
V = \frac{1}{3} \pi r^2 h
\]

Since the diameter of the pile is three times the height and since the diameter is twice the radius we have

\[
2r = 3h \text{ or } r = \frac{3}{2} h
\]

substituting for \( r \) we obtain an equation relating only \( V \) and \( h \):

\[
V = \frac{1}{3} \pi \left(\frac{3}{2} h\right)^2 h = \frac{3}{4} \pi h^3
\]

d. Implicitly differentiate both sides of equation with respect to time, \( t \).

\[
\frac{dV}{dt} = \frac{3}{4} \pi \left(3h^2 \frac{dh}{dt}\right) = \frac{9}{4} \pi h^2 \frac{dh}{dt}
\]

or solving for \( dh/dt \):

\[
\frac{dh}{dt} = \frac{4}{9\pi h^2} \frac{dV}{dt}
\]
e. Substitute given information into resulting equation and solve for the unknown rate.
\[ \frac{dh}{dt} \bigg|_{h=15} = \frac{4}{9\pi(15)^2} (10) = \frac{8}{405\pi} \approx 0.00629 \text{ ft/min} \]

4. A trough is 12 feet long and 3 feet across the top (see figure below). Its ends are isosceles triangles with altitudes of 3 feet. If water is being pumped into the trough at 2 cubic feet per minute, how fast is the water level rising when the water in the trough is 1 foot deep?

![Diagram of the trough with dimensions labeled: 3 feet height, 12 feet length, 3 feet width, and arrows indicating water flow.]

a. Assign symbols to all quantities (there are THREE in this case) that are functions of time. Let \( t \) be the time in minutes.

The water will take the shape of the trough. Let \( h \) be the height of the water in the trough and let \( w \) be the width across the top of the water (see diagram below). Let \( V \) be the volume of the water.

![Diagram of the trough with symbols labeled: \( w \), \( h \), and \( V \).]

b. Write the given information & the rate you are trying to find as appropriate derivatives.

Given:
\[ \frac{dV}{dt} = 2 \text{ ft}^3/\text{min} \]

Find:
\[ \frac{dh}{dt} \text{ when } h = 1 \text{ ft} \]

c. Write an equation that relates the various quantities of the problem. Use the geometry of the situation to eliminate one of the variables by substitution so that equation has only two variables.

(HINT: Use similar triangles to get a relationship between two of your variables & then substitute to eliminate one of the three variables.)

First we have
\[ V = \frac{1}{2} wh(12) = 6wh \]

But, by similar triangles we have
\[ \frac{w}{h} = \frac{3}{3} \implies w = h \]

so that
\[ V = 6h^2 \]

d. Implicitly differentiate both sides of equation with respect to time, \( t \).
\[ \frac{dV}{dt} = 12h \frac{dh}{dt} \]

or
\[ \frac{dh}{dt} = \frac{1}{12h} \frac{dV}{dt} \]
e. Substitute given information into resulting equation and solve for the unknown rate.

\[
\left. \frac{dh}{dt} \right|_{h=1} = \frac{1}{12(1)} (2) = \frac{1}{6} \text{ ft/min} \approx 0.167 \text{ ft/min}
\]

5. Carefully review each solution. Ask if you have questions.

a. Let \( y \) be the height of the top of the ladder above ground and \( x \) be the distance from the house to the bottom of the ladder. Then we are given that \( \frac{dx}{dt} = 2 \) and asked to find \( \frac{dy}{dt} \) when \( y = 12, 5, \) & 1. The ladder and house form a right triangle, so by the Pythagorean Theorem we have

\[
x^2 + y^2 = 13^2
\]

We differentiate implicitly with respect to \( t \)

\[
\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(13^2)
\]

\[
\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = 0
\]

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]

Solving for \( \frac{dy}{dt} \) yields

\[
2y \frac{dy}{dt} = -2x \frac{dx}{dt}
\]

\[
\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}
\]

Plugging in our given value yields

\[
\frac{dy}{dt} = -\frac{x}{y} (2) = -\frac{2x}{y}
\]

Now when \( y = 12 \), by the Pythagorean Theorem we have

\[
x^2 + 12^2 = 13^2 \Rightarrow x^2 = 169 - 144 \Rightarrow x^2 = 25 \Rightarrow x = 5
\]

Thus when the top of the ladder is 12 feet above the ground,

\[
\frac{dy}{dt} = -\frac{2(5)}{12} = -\frac{5}{6} \approx -0.83 \text{ ft/s}
\]

Note that the negative sign indicates that \( y \) is decreasing (i.e. the top of the ladder is moving downward). We can similarly find \( \frac{dy}{dt} \) for when \( y = 5 \) & 1. The table below summarizes the findings.

<table>
<thead>
<tr>
<th>( y ) (in feet)</th>
<th>12</th>
<th>5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dy}{dt} ) (in ft/s)</td>
<td>-0.83</td>
<td>-4.80</td>
<td>-25.92</td>
</tr>
</tbody>
</table>

Note that the rate at which the ladder is sliding down the side of the house is NOT constant. It appears to be speeding up. To express the rate at which the ladder is sliding down the side of the house as a function of the height the top of the ladder, we note that at any instant

\[
x^2 + y^2 = 13^2 \Rightarrow x^2 = 169 - y^2 \Rightarrow x = \sqrt{169 - y^2}
\]

Thus

\[
\frac{dy}{dt} = -\frac{2x}{y} = -\frac{2\sqrt{169 - y^2}}{y}
\]

Since \( y > 0, \sqrt{y^2} = y \) and we can further simplify this as
\[
\frac{dy}{dt} = -2 \sqrt{\frac{169 - y^2}{y^2}} = -2 \sqrt{\frac{169}{y^2} - 1}
\]

Note that
\[
\lim_{y \to 0^+} \frac{dy}{dt} = \lim_{y \to 0^+} -2 \sqrt{\frac{169}{y^2} - 1} = -\infty
\]

since
\[
\frac{169}{y^2} \to \infty \text{ as } y \to 0^+
\]

Thus, as the top of the ladder gets closer and closer to the ground, the speed at which the ladder is falling approaches (negative) infinity. Note that in real life friction will limit the speed of the falling ladder. Our model does not include the effects of friction.

b. Let \( A \) be the enclosed area. Using the same variables from part (a), we want to find \( \frac{dA}{dt} \) when \( y = 12, 5, \& 1 \). Note that the region enclosed is a right triangle so that
\[
A = \frac{1}{2} xy
\]

Differentiating implicitly with respect to \( t \), we have
\[
\frac{d}{dt} (A) = \frac{d}{dt} \left( \frac{1}{2} xy \right)
\]
\[
\frac{dA}{dt} = \frac{1}{2} \left( x \frac{dy}{dt} + y \frac{dx}{dt} \right)
\]

Using the results from part (a), we have

<table>
<thead>
<tr>
<th>( y ) (in ft)</th>
<th>( x ) (in feet)</th>
<th>( \frac{dy}{dt} ) (in ft/s) (from part (a))</th>
<th>( \frac{dA}{dt} ) (in ft^2/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12 ft</td>
<td>5</td>
<td>-0.83</td>
<td>9.9</td>
</tr>
<tr>
<td>5 ft</td>
<td>12</td>
<td>-4.80</td>
<td>-23.8</td>
</tr>
<tr>
<td>1 ft</td>
<td>12.96</td>
<td>-25.92</td>
<td>-167.0</td>
</tr>
</tbody>
</table>

c. Let \( \theta \) be the angle between the ladder and the ground. We are asked to find \( \frac{d\theta}{dt} \) when \( y = 12, 5, \& 1 \).

Using the variables introduced above we have
\[
\sin \theta = \frac{y}{13}
\]

Differentiating implicitly with respect to \( t \), we have
\[
\frac{d}{dt} (\sin \theta) = \frac{d}{dt} \left( \frac{y}{13} \right)
\]
\[
\cos \theta \frac{d\theta}{dt} = \frac{1}{13} \frac{dy}{dt}
\]
\[
\frac{d\theta}{dt} = \frac{1}{13} \frac{1}{\cos \theta} \frac{dy}{dt} = \frac{\sec \theta}{13} \frac{dy}{dt}
\]

Note that
\[
\sec \theta = \frac{13}{x}
\]

So that
\[
\frac{d\theta}{dt} = \frac{\sec \theta}{13} \frac{dy}{dt} = \frac{1}{x} \frac{dy}{dt}
\]

Thus
Let \( x \) be the distance from the blue car to where the red car was at noon. Let \( y \) the distance the red car has traveled since noon.

If \( s \) is the distance between the cars then we are asked to find \( \frac{ds}{dt} \) at 2 & 4 p.m. given that \( \frac{dx}{dt} = -40 \text{ mph} \) (notice that the distance \( x \) is getting smaller) and \( \frac{dy}{dt} = 25 \text{ mph} \). The Pythagorean Theorem equates all our quantities:

\[
x^2 + y^2 = s^2
\]

Differentiating implicitly with respect to \( t \), we have

\[
\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(s^2)
\]

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2s \frac{ds}{dt}
\]

Thus

\[
\frac{ds}{dt} = \frac{1}{s} \left( x \frac{dx}{dt} + y \frac{dy}{dt} \right)
\]

At 2 p.m. \( x = 120 \) (blue car has traveled 80 miles, leaving 120 miles to the location of red car at noon) and \( y = 50 \) so that

\[
s = \sqrt{120^2 + 50^2} = 130
\]

Hence

\[
\frac{ds}{dt} = \frac{1}{130} (120(-40) + 50(25)) \approx -27.31 \text{ mph}
\]

At 4 p.m. \( x = 40 \) (blue car has traveled 160 miles, leaving 40 miles to the location of red car at noon) and \( y = 100 \) so that

\[
s = \sqrt{40^2 + 100^2} = 20\sqrt{29} \approx 107.7
\]

Hence
\[
\frac{ds}{dt} = \frac{1}{20\sqrt{29}}(40(-40) + 100(25)) \approx 8.36 \text{ mph}
\]

Note that the distance between the cars was decreasing at 2 p.m. (ds/dt was negative) and increasing at 4 p.m. (ds/dt positive).