Quiz #14 Solutions

PQ's

#1

![Cube Diagram]

\[ V = \text{Volume of Cube} \]

**Given:** \( \frac{ds}{dt} = 0.5 \text{ cm/s} \)

**Unknown:** \( \frac{dv}{dt} \)

#3

![Cylinder Diagram]

**Given:** \( \frac{dv}{dt} = 2 \text{ cm}^3/\text{min} \)

**Find:** \( \frac{dh}{dt} \)

#4

\[ \frac{dh}{dt} = 1 \text{ cm/min} \Rightarrow \text{Find:} \ \frac{dv}{dt} \]

Exercises:

#1

![Rectangular Tank Diagram]

**Volume of Water in Tank**

**Given:** \( \frac{dv}{dt} = 0.7 \text{ ft}^3/\text{min} \)

**Find:** \( \frac{dh}{dt} \)

\[ V = 18h \Rightarrow \frac{d}{dt}(V) = \frac{d}{dt}(18h) \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{18} \frac{dv}{dt} \]

\[ \frac{dv}{dt} = 18 \frac{dh}{dt} \]

\[ \frac{1}{18} (0.7) \text{ ft/min} \]
Let \( V \) = volume of cone

Given: \( \frac{dr}{dt} = \frac{dh}{dt} = 2 \text{ cm/s} \)

Find: \( \frac{dV}{dt} \) when \( r = 10 \text{ cm} \) \( , h = 20 \text{ cm} \)

\[ V = \frac{1}{3} \pi r^2 h \]

\[
\frac{dV}{dt} = \frac{1}{3} \pi \left( \frac{d}{dt} (r^2 h) \right) \quad \text{(Product Rule)}
\]

\[
\frac{dV}{dt} = \frac{1}{3} \pi \left( 2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right) \quad \text{(Chain Rule)}
\]

So

\[
\left. \frac{dV}{dt} \right|_{r=10, \ h=20} = \frac{1}{3} \pi \left( \frac{2(10)(10)(20)}{800} + \frac{10^2(2)}{200} \right)
\]

\[
= \left( \frac{1000 \pi}{3} \right) \text{ cm}^3/\text{s}
\]
(a) Given: \( \frac{dk}{dt} = 800 \text{ km/h} \)

Find: \( \frac{dl}{dt} \)  

Half an hour after plane passes over station (some versions Half a minute)

By Pythag. Thm: \( x^2 + 6^2 = l^2 \)

\[ \frac{d}{dt} (x^2 + 6^2) = \frac{d}{dt} (l^2) \]

\[ 2x \frac{dk}{dt} + 0 = 2l \frac{dl}{dt} \]

So

\[ \frac{dl}{dt} = \frac{x}{l} \frac{dk}{dt} \]

So

\[ \frac{dl}{dt} = \frac{800}{6} \frac{dk}{dt} = 0 \text{ km/h} \]

(b) Plane is directly above station when

\( x = 0 \) and \( l = 6 \text{ km} \)

So

\[ \frac{dl}{dt} = \frac{0}{6} \frac{dk}{dt} = 0 \text{ km/h} \]

Use either of these values to find \( \frac{dl}{dt} \)
Given: \( \frac{dh}{dt} = 1200 \text{ km/h} \)  
Find: \( \frac{d\theta}{dt} \)  
3 min. after lift-off.

\[
\tan \theta = \frac{h}{16} \quad \Rightarrow \quad \frac{d}{dt}(\tan \theta) = \frac{d}{dt} \left( \frac{h}{16} \right)
\]

\[
\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{16} \frac{dh}{dt}
\]

\[
\Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{16 \cos^2 \theta} \frac{dh}{dt} \quad (\text{since } \sec^2 \theta = \frac{1}{\cos^2 \theta})
\]

\[
h = \frac{1200 \text{ km}}{\theta} \cdot \left( \frac{3}{60 \text{ h}} \right) = 60 \text{ km}
\]

\[
h^2 + 16^2 = \ell^2
\]

\[
\ell = \sqrt{60^2 + 16^2}
\]

\[
\left. \frac{d\theta}{dt} \right|_{\text{after } 3 \text{ mins.}} = \frac{1}{16} \cdot \frac{16}{60^2 + 16^2} \cdot 1200 = \frac{(16)(1200)}{60^2 + 16^2} \text{ rad/h}
\]
SECTION 3.11 | Related Rates 199

**Step 3.** Use the given data to find the unknown derivative.
Apply Eq. (5) with \( x = 5 \) and \( \frac{dx}{dt} = 3 \). The bale is rising at the rate
\[
\frac{dh}{dt} = \frac{x \frac{dx}{dt}}{\sqrt{x^2 + 4.5^2}} = \frac{(5)(3)}{\sqrt{5^2 + 4.5^2}} \approx 2.23 \text{ m/s}
\]

**3.11 SUMMARY**

- Related-rate problems present us with situations in which two or more variables are related and we are asked to compute the rate of change of one of the variables in terms of the rates of change of the other variable(s).
- Draw a diagram if possible. It may also be useful to break the solution into three steps:

  **Step 1.** Assign variables and restate the problem.

  **Step 2.** Find an equation that relates the variables and differentiate.

  This gives us an equation relating the known and unknown derivatives. Remember not to substitute values for the variables until after you have computed all derivatives.

  **Step 3.** Use the given data to find the unknown derivative.

- The two facts from geometry arise often in related-rate problems: Pythagorean Theorem and the Theorem of Similar Triangles (ratios of corresponding sides are equal).

**3.11 EXERCISES**

**Preliminary Questions**

1. Assign variables and restate the following problem in terms of known and unknown derivatives (but do not solve it): How fast is the volume of a cube increasing if its side increases at a rate of 0.5 cm/s?
2. What is the relation between \( dV/dt \) and \( dr/dt \) if \( V = \left(\frac{4}{3}\right)\pi r^3 \)?

In Questions 3 and 4, water pours into a cylindrical glass of radius 4 cm. Let \( V \) and \( h \) denote the volume and water level respectively, at time \( t \).

3. Restate this question in terms of \( dV/dt \) and \( dh/dt \): How fast is the water level rising if water pours in at a rate of 2 cm³/min?
4. Restate this question in terms of \( dV/dt \) and \( dh/dt \): At what rate is water pouring in if the water level rises at a rate of 1 cm/min?

**Exercises**

In Exercises 1 and 2, consider a rectangular bathtub whose base is 18 ft².

1. How fast is the water level rising if water is filling the tub at a rate of 0.7 ft³/min?
2. At what rate is water pouring into the tub if the water level rises at a rate of 0.8 ft/min?
3. The radius of a circular oil slick expands at a rate of 2 m/min. 
   (a) How fast is the area of the oil slick increasing when the radius is 23 m?
   (b) If the radius is 0 at time \( t = 0 \), how fast is the area increasing after 3 min?
4. At what rate is the diagonal of a cube increasing if its edges are increasing at a rate of 2 cm/s?

In Exercises 5–8, assume that the radius \( r \) of a sphere is expanding at a rate of 30 cm/min. The volume of a sphere is \( V = \frac{4}{3}\pi r^3 \) and its surface area is \( 4\pi r^2 \). Determine the given rate.

5. Volume with respect to time when \( r = 15 \) cm.
6. Volume with respect to time at \( t = 2 \) min, assuming that \( r = 0 \) at \( t = 0 \).
7. Surface area with respect to time when \( r = 40 \) cm.
8. Surface area with respect to time at \( t = 2 \) min, assuming that \( r = 10 \) at \( t = 0 \).
In Exercises 9–12, refer to a 5-meter ladder sliding down a wall, as in Figures 1 and 2. The variable \( h \) is the height of the ladder’s top at time \( t \), and \( x \) is the distance from the wall to the ladder’s bottom.

9. Assume the bottom slides away from the wall at a rate of 0.8 m/s. Find the velocity of the top of the ladder at \( t = 2 \) s if the bottom is 1.5 m from the wall at \( t = 0 \) s.

10. Suppose that the top is sliding down the wall at a rate of 1.2 m/s. Calculate \( \frac{dx}{dt} \) when \( h = 3 \) m.

11. Suppose that \( h(0) = 4 \) and the top slides down the wall at a rate of 1.2 m/s. Calculate \( x \) and \( \frac{dx}{dt} \) at \( t = 2 \) s.

12. What is the relation between \( h \) and \( x \) at the moment when the top and bottom of the ladder move at the same speed?

13. A conical tank has height 3 m and radius 2 m at the top. Water flows in at a rate of 2 m³/min. How fast is the water level rising when it is 2 m? At what rate is water flowing in?

14. Follow the same set-up as Exercise 13, but assume that the water level is rising at a rate of 0.3 m/min when it is 2 m. What rate is water flowing in?

15. The radius \( r \) and height \( h \) of a circular cone change at a rate of 2 cm/s. How fast is the volume of the cone increasing when \( r = 10 \) and \( h = 20 \) cm?

16. A road perpendicular to a highway leads to a farmhouse located 2 km away (Figure 8). An automobile travels past the farmhouse at a speed of 80 km/h. How fast is the distance between the automobile and the farmhouse increasing when the automobile is 6 km past the intersection of the highway and the road?

17. A man of height 1.8 meters walks away from a 5-meter lamppost at a speed of 1.2 m/s (Figure 9). Find the rate at which his shadow is increasing in length.

18. As Claudia walks away from a 264-cm lamppost, the tip of her shadow moves twice as fast as she does. What is Claudia’s height?

19. At a given moment, a plane passes directly above a radar station at an altitude of 6 km.

(a) The plane’s speed is 800 km/h. How fast is the distance between the plane and the station changing half an hour later?

(b) How fast is the distance between the plane and the station changing when the plane passes directly above the station?

20. In the setting of Exercise 19, let \( \theta \) be the angle that the line through the radar station and the plane makes with the horizontal. How fast is \( \theta \) changing 12 min after the plane passes over the radar station?

21. A hot air balloon rising vertically is tracked by an observer located 4 km from the lift-off point. At a certain moment, the angle between the observer’s line of sight and the horizontal is \( \frac{\pi}{6} \), and it is changing at a rate of 0.2 rad/min. How fast is the balloon rising at this moment?

22. A laser pointer is placed on a platform that rotates at a rate of 20 revolutions per minute. The beam hits a wall 8 m away, producing a dot of light that moves horizontally along the wall. Let \( \theta \) be the angle between the beam and the line through the searchlight perpendicular to the wall (Figure 10). How fast is this dot moving when \( \theta = \frac{\pi}{6} \) ?

![Figure 10](image)

23. A rocket travels vertically at a speed of 1,200 km/h. The rocket is tracked through a telescope by an observer located 16 km from the launching pad. Find the rate at which the angle between the telescope and the ground is increasing 3 min after lift-off.

24. Using a telescope, you track a rocket that was launched 4 km away, recording the angle \( \theta \) between the telescope and the ground at half-second intervals. Estimate the velocity of the rocket if \( \theta(10) = 0.205 \) and \( \theta(10.5) = 0.225 \).

25. A police car traveling south toward Sioux Falls at 160 km/h pursues a truck traveling east away from Sioux Falls, Iowa, at 140 km/h (Figure 11). At time \( t = 0 \), the police car is 20 km north and the truck is 30 km east of Sioux Falls. Calculate the rate at which the distance between the vehicles is changing:

(a) At time \( t = 0 \)

(b) 5 minutes later

![Figure 11](image)
26. A car travels down a highway at 25 m/s. An observer stands 150 m from the highway.
(a) How fast is the distance from the observer to the car increasing when the car passes in front of the observer? Explain your answer without making any calculations.
(b) How fast is the distance increasing 20 s later?

27. In the setting of Example 5, at a certain moment, the tractor’s speed is 3 m/s and the bale is rising at 2 m/s. How far is the tractor from the bale at this moment?

28. Placido pulls a rope attached to a wagon through a pulley at a rate of q m/s. With dimensions as in Figure 12:
(a) Find a formula for the speed of the wagon in terms of q and the variable x in the figure.
(b) Find the speed of the wagon when x = 0.6 if q = 0.5 m/s.

29. Julian is jogging around a circular track of radius 50 m. In a coordinate system with origin at the center of the track, Julian's x-coordinate is changing at a rate of -1.25 m/s when his coordinates are (40, 30). Find dy/dt at this moment.

30. A particle moves counterclockwise around the ellipse with equation $9x^2 + 16y^2 = 25$ (Figure 13).
(a) In which of the four quadrants is dx/dt > 0? Explain.
(b) Find a relation between dx/dt and dy/dt.
(c) At what rate is the x-coordinate changing when the particle passes the point $(1, 1)$ if its y-coordinate is increasing at a rate of 6 m/s?
(d) Find dy/dt when the particle is at the top and bottom of the ellipse.

32. Find b if $P = 25$ kPa, $dP/dt = 12$ kPa/min, $V = 100$ cm$^3$, and $dV/dt = 20$ cm$^3$/min.

33. The base x of the right triangle in Figure 14 increases at a rate of 5 cm/s, while the height remains constant at $h = 20$. How fast is the angle $\theta$ changing when $x = 20$?

34. Two parallel paths 15 m apart run east-west through the woods. Brooke jogs east on one path at 10 km/h, while Jamail walks west on the other path at 6 km/h. If they pass each other at time $t = 0$, how far apart are they 3 s later, and how fast is the distance between them changing at that moment?

35. A particle travels along a curve $y = f(x)$ as in Figure 15. Let $L(t)$ be the particle’s distance from the origin.
(a) Show that $\frac{dL}{dt} = \left(x + f(x) \frac{f'(x)}{\sqrt{x^2 + f(x)^2}}\right) \frac{dx}{dt}$ if the particle’s location at time $t$ is $P = (x, f(x))$.
(b) Calculate $L'(t)$ when $x = 1$ and $x = 2$ if $f(x) = \sqrt{3x^2 - 8x + 9}$ and $dx/dt = 4$.

36. Let $\theta$ be the angle in Figure 15, where $P = (x, f(x))$. In the setting of the previous exercise, show that
$$\frac{d\theta}{dt} = \left(\frac{xf'(x) - f(x)}{x^2 + f(x)^2}\right) \frac{dx}{dt}$$

**Hint:** Differentiate $\tan \theta = f(x)/x$ and observe that $\cos \theta = x/\sqrt{x^2 + f(x)^2}$.

Exercises 37 and 38 refer to the baseball diamond (a square of side 90 ft) in Figure 16.

37. A baseball player runs from home plate toward first base at 20 ft/s. How fast is the player’s distance from second base changing when the player is halfway to first base?