Mixing Problems
An application of Differential Equations (Section 7.3)

A typical mixing problem investigates the behavior of a mixed solution of some substance. Typically the solution is being mixed in a large tank or vat. A solution (or solutions) of a given concentration enters the mixture at some fixed rate and is thoroughly mixed in the tank or vat. The tank or vat is also being drained at some fixed rate. Setting up a model for this situation typically involves a differential equation. If $y(t)$ is the amount of the substance in the tank or vat at time $t$, then $\frac{dy}{dt}$ is the rate the substance is being added minus the rate at which it is being removed.

$$\frac{dy}{dt} = \left(\text{rate in}\right) - \left(\text{rate out}\right)$$

Example:

A large vat holds 100 gallons of water that is to be mixed with sugar and then used to make soft drinks. Sugar-water containing 5 tablespoons of sugar per gallon enters the vat through a pipe at a rate of 2 gallons per minute. Another pipe pumps sugar-water with 10 tablespoons of sugar per gallon into the vat at a rate of 1 gallon per minute. The vat is kept well mixed, so that the concentration of sugar in the vat is essentially uniform. Sugar-water is drained out of the vat at a rate of 3 gallons per minute. Notice that the amount flowing into the vat is the same as the amount flowing out of the vat, so there are always 100 gallons of liquid in the vat. Find the amount of sugar in the vat at time $t$ if the vat initially has 900 tablespoons in it.

Solution: Let $y(t)$ be the amount of sugar in the vat at time $t$. Since there are initially 900 tablespoons of sugar in the vat, we have $y(0) = 900$. The amount of sugar flowing into the vat is the product of the concentration of the entering solution and its flow rate. Hence, through the first pipe we have

$$5 \frac{\text{tblsp}}{\text{gal}} \times 2 \frac{\text{gal}}{\text{min}} = 10 \frac{\text{tblsp}}{\text{min}}$$

entering the vat and through the second pipe we have

$$10 \frac{\text{tblsp}}{\text{gal}} \times 1 \frac{\text{gal}}{\text{min}} = 10 \frac{\text{tblsp}}{\text{min}}$$

Hence, the net amount of sugar entering the vat is

$$10 \frac{\text{tblsp}}{\text{min}} + 10 \frac{\text{tblsp}}{\text{min}} = 20 \frac{\text{tblsp}}{\text{min}}$$

The amount of sugar leaving the vat at any given moment depends on the concentration of sugar in the vat at that moment. The amount of sugar in the vat at time $t$ is $y(t)$. Since the tank always contains 100 gallons of liquid, the concentration of sugar is given by $y/100$. Thus, the sugar leaving the vat is

$$\frac{y}{100} \frac{\text{gal}}{\text{gal}} \times 3 \frac{\text{gal}}{\text{min}} = \frac{3y}{100} \frac{\text{tblsp}}{\text{min}}$$

Hence the rate of change of the sugar in the vat (in tblsp/min) is

$$\frac{dy}{dt} = \left(\text{rate in}\right) - \left(\text{rate out}\right) = 20 - \frac{3y}{100}$$

Thus, to find the amount of sugar in the vat at time $t$ we want to solve the initial value problem

$$\frac{dy}{dt} = 20 - \frac{3y}{100}, \quad y(0) = 900$$

Using separation of variables, we obtain the following solution

$$y(t) = \frac{700}{3} e^{-0.03t} + \frac{2000}{3}$$

What is the equilibrium solution for this differential equation and what is its significance?